Limits of sequences

Definition 1 (Bounded sequence)**.** *We say that a sequence a_n is bounded if there exists real number M such that $|a_n| \leq M$ for any natural number $n \in \mathbb{N}$.*

Definition 2 (Convergent sequence)**.** *We say that a sequence a_n is convergent if there exists real number L such that for any $\varepsilon > 0$, there exists natural number $N \in \mathbb{N}$ such that if $n > N$, then*

$$|a_n - L| < \varepsilon.$$

In this case we say that the limit of a_n is L and write

$$\lim_{n\to\infty} a_n = L.$$

Theorem 3. *If a_n is convergent, then a_n is bounded.*

Proof. Suppose $\lim_{n\to\infty} a_n = L$. Then there exists $N \in \mathbb{N}$ such that if $n > N$, then $|a_n - L| < 1$ which implies $|a_n| \leq |L| + 1$. Take

$$M = \max\{|a_0|, |a_1|, |a_2|, \cdots, |a_n|, |L| + 1\}.$$

Then we have $|a_n| \leq M$ for any $n \in \mathbb{N}$. □

Theorem 4. *Suppose $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Then $\lim_{n\to\infty} a_n b_n = ab$.*

Proof. Since a_n and b_n are convergent, they are bounded and there exists real numbers $M_1, M_2 > 0$ such that $|a_n| \leq M_1$ and $|b_n| \leq M_2$ for any $n \in \mathbb{N}$. Note that we have $|a| \leq M_1$ and $|b| \leq M_2$. Now for any $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that if $n > N_1$, we have

$$|a_n - a| < \frac{\varepsilon}{2M_2}$$

and there exists $N_2 \in \mathbb{N}$ such that if $n > N_2$, we have

$$|b_n - b| < \frac{\varepsilon}{2M_1}.$$

Now take $N = \max\{N_1, N_2\}$. Then if $n > N$, we have

$$\begin{aligned}
|a_n b_n - ab| &= |a_n b_n - ab_n + ab_n - ab| \\
&\leq |(a_n - a)b_n| + |a(b_n - b)| \\
&\leq |a_n - a||b_n| + |a||b_n - b| \\
&< \left(\frac{\varepsilon}{2M_2}\right) M_2 + M_1 \left(\frac{\varepsilon}{2M_1}\right) \\
&= \varepsilon.
\end{aligned}$$

Therefore we conclude that $\lim_{n\to\infty} a_n b_n = ab$. □

Theorem 5. *If* $\lim_{n\to\infty} a_n = 0$, *then* $\lim_{n\to\infty} |a_n| = 0$.

Proof. Suppose $\lim_{n\to\infty} a_n = 0$. For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n > N$, then $|a_n - 0| < \varepsilon$. Now if $n > N$, we have $||a_n| - 0| = |a_n| = |a_n - 0| < \varepsilon$. Therefore we conclude that $\lim_{n\to\infty} |a_n| = 0$. □

Theorem 6. *If* $a_n \geq 0$ *for any* n *and* $\lim_{n\to\infty} a_n = a$, *then* $\lim_{n\to\infty} \sqrt{a_n} = \sqrt{a}$.

Proof. Suppose $a = 0$. Then for any $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that if $n > N$, then $|a_n| < \varepsilon^2$. Now if $n > N$, we have $|\sqrt{a_n}| = \sqrt{a_n} < \varepsilon$. Therefore we conclude that $\lim_{n\to\infty} \sqrt{a_n} = 0$.

Suppose $a > 0$. For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n > N$, then $|a_n - a| < \varepsilon\sqrt{a}$. It follows that if $n > N$, then

$$\begin{aligned}
|\sqrt{a_n} - \sqrt{a}| &= \frac{|\sqrt{a_n} - \sqrt{a}||\sqrt{a_n} + \sqrt{a}|}{|\sqrt{a_n} + \sqrt{a}|} \\
&= \frac{|a_n - a|}{|\sqrt{a_n} + \sqrt{a}|} \\
&< \frac{\varepsilon\sqrt{a}}{\sqrt{a}} \\
&= \varepsilon.
\end{aligned}$$

We conclude that $\lim_{n\to\infty} \sqrt{a_n} = \sqrt{a}$. □