Week 1

MATH 2040B

September 15, 2020

1 Concepts

1. Vector spaces (Class Note 1)
2. Subspaces (Class Note 2)

2 Notations

1. Let $A = \begin{pmatrix} u & v \\ w & z \end{pmatrix}$ be a complex matrix, then

   (a) $\mathrm{tr}(A) = u + z$

   (b) $\det(A) = uz - vw$

   (c) $A^* = \begin{pmatrix} \bar{u} & \bar{w} \\ \bar{v} & \bar{z} \end{pmatrix}$, where $\bar{u}$ is the complex conjugate of u

3 Problems

1. Let $\mathrm{Mat}_{2\times 2}(\mathbb{C})$ be the set of 2×2 complex matrices, and $u_2 \subset \mathrm{Mat}_{2\times 2}(\mathbb{C})$ be the subset of skew symmetric matrices, i.e.

$$u_2 = \{A \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) : A^* + A = 0\}$$

(a) Show that $\mathrm{Mat}_{2\times 2}(\mathbb{C})$ with the usual matrix addition and scalar multiplication forms a complex vector space. Is it also a real vector space?

(b) Show that u_2 is a real subspace of $\mathrm{Mat}_{2\times 2}(\mathbb{C})$, is it also a complex subspace of $\mathrm{Mat}_{2\times 2}(\mathbb{C})$?

2. Let

$$\begin{aligned} U &= \left\{A \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) : A^2 = \mathrm{tr}(A)A\right\} \\ V &= \left\{A \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) : A^2 + \det(A)I = 0\right\} \end{aligned}$$

(a) Is U a vector subspace of $\mathrm{Mat}_{2\times 2}(\mathbb{C})$?

(b) Is V a vector subspace of $\mathrm{Mat}_{2\times 2}(\mathbb{C})$?

3. Let V be a vector space over $\mathbb{R}$, and W_1, W_2 be vector subspaces of V.

   (a) Show by example that $W_1 \cup W_2$ may not be a vector subspace of V.

   (b) Show that if $W_1 \cup W_2$ is a vector subspace, then either $W_1 \subset W_2$, or $W_2 \subset W_1$

   (c) If
$$V = \bigcup_{i=1}^{n} U_i$$
   for some vector subspaces $U_1, \ldots, U_n$ of V, show that $U_i = V$ for some i.