Week 8

MATH 2040

November 4, 2020

1 Problems

1. (a) Let V be an m-dimension vector space, $T : V \to V$ be a linear transformation and β be a basis of V. Suppose $f(t) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_0$, then we can define f on linear transformation and on matrix such that $f(T) = a_n T^n + a_{n-1}T^{n-1} + \cdots + a_0 I_v$ and $f([T]_\beta) = a_n[T]_\beta^n + a_{n-1}[T]_\beta^{n-1} + \cdots + a_0 I$. Show that $[f(T)]_\beta = f([T]_\beta)$.

   (b) Show that for some invertible Q, $f(Q^{-1}[T]_\beta Q) = Q^{-1}f([T]_\beta)Q$.

   Ans: Recall that for all linear transformation $T_1, T_2 : U \to V$, β, γ are basis of U, V, and $\alpha \in F$, we have

   - $[T_1 + T_2]_\beta^\gamma = [T_1]_\beta^\gamma + [T_2]_\beta^\gamma$
   - when $U = V$ and $\beta = \gamma$, $[T_1T_2]_\beta = [T_1]_\beta[T_2]_\beta$
   - $[\alpha T_1]_\beta^\gamma = \alpha[T_1]_\beta^\gamma$.

   So in this problem, we have that $[T^k]_\beta = [T]_\beta^k$ and $[I_v]_\beta = I$.

   (a) $[f(T)]_\beta = [\sum_{i=1}^n a_i T^i]_\beta = \sum_{i=1}^n [a_i T^i]_\beta = \sum_{i=1}^n a_i[T^i]_\beta = \sum_{i=1}^n a_i[T]_\beta^i = f([T]_\beta)$

   (b) Here I use 2 method to show that

      i. Since Q is invertible, there exists some basis γ of V such that $Q = [I]_\gamma^\beta$, then

      $$f(Q^{-1}[T]_\beta Q) = f([I]_\beta^\gamma[T]_\beta[I]_\gamma^\beta) = f([T]_\gamma) = [f(T)]_\gamma = [I]_\beta^\gamma[f(T)]_\beta[I]_\gamma^\beta = Q^{-1}[f(T)]_\beta Q$$

      ii. For all i, we have that $(Q^{-1}[T]_\beta Q)^i = Q^{-1}[T]_\beta^i Q$, so

      $$f(Q^{-1}[T]_\beta Q) = \sum_{i=0}^n a_i(Q^{-1}[T]_\beta Q)^i = \sum_{i=0}^n a_i Q^{-1}[T]_\beta^1 Q = Q^{-1}(\sum_{i=0}^n a_i[T]_\beta^1)Q = Q^{-1}[f(T)]_\beta Q$$

2. If $\{\lambda_1, \lambda_2, \cdots, \lambda_k\}$ is the set of eigenvalues of linear transformation $T : V \to V$ and $f(t) = (t-\lambda_1)(t-\lambda_2)\cdots(t-\lambda_k)$. Suppose that T is diagonalizable, show that $f(T) = 0$.

   Ans: Since T is diagonalizable, we can find a basis $\beta = \{v_1, v_2, \cdots, v_n\}$ of V, where v_i are all eigenvectors of T, $n \geq k$. Note that $\mu_T(\lambda_i) = n_i$, reorder β and we can get

$$[T]_\beta = \begin{pmatrix} \lambda_1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & \lambda_1 & & & & & & \\ & & & \lambda_2 & & & & & \\ & & & & \ddots & & & & \\ & & & & & \lambda_2 & & & \\ & & & & & & \ddots & & \\ & & & & & & & \lambda_k & \\ & & & & & & & & \ddots & \\ & & & & & & & & & \lambda_k \end{pmatrix} = \begin{pmatrix} \lambda_1 I_{n_1} & & & \\ & \lambda_2 I_{n_2} & & \\ & & \ddots & \\ & & & \lambda_k I_{n_k} \end{pmatrix}.$$

Then $$[T]_\beta - \lambda_i I = \begin{pmatrix} (\lambda_1-\lambda_i)I_{n_1} & & & & & & \\ & \ddots & & & & & \\ & & (\lambda_{i-1}-\lambda_i)I_{n_{i-1}} & & & & \\ & & & 0 & & & \\ & & & & (\lambda_{i+1}-\lambda_i)I_{n_{i+1}} & & \\ & & & & & \ddots & \\ & & & & & & (\lambda_k-\lambda_i)I_{n_k} \end{pmatrix},$$

so we can think $[T]_\beta - \lambda_i I$ is a $k \times k$ matrix with i col i row is zero. Thereforce

$$[f(T)]_\beta = [(T-\lambda_1 I)\cdots(T-\lambda_k I)]_\beta = ([T]_\beta - \lambda_1 I)\cdots([T]_\beta - \lambda_k I) = 0$$

then $f(T) = 0$ since we know that the map $\Phi : \mathcal{L}(V,V) \to M_{n\times n}(\mathbb{F})$, $\Phi(T) = [T]_\beta$ is injective.

3. Let $A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$

(a) Find the characteristic polynomial of A.

(b) Find eigenvectors.

(c) Is A diagonalizable? If so, find $Q \in M_{3\times 3}(\mathbb{R})$ such that $Q^{-1}AQ$ is diagonal.

Ans:

(a)

$$\begin{aligned} f_A(t) &= \begin{vmatrix} 3-t & 1 & 1 \\ 2 & 4-t & 2 \\ -1 & -1 & 1-t \end{vmatrix} = \begin{vmatrix} 3-t & 1 & 1 \\ 2 & 4-t & 2 \\ 4-t & 4-t & 4-t \end{vmatrix} \\ &= (4-t)\begin{vmatrix} 3-t & 1 & 1 \\ 2 & 4-t & 2 \\ 1 & 1 & 1 \end{vmatrix} = (4-t)\begin{vmatrix} 2-t & 0 & 0 \\ 0 & 2-t & 0 \\ 1 & 1 & 1 \end{vmatrix} = (4-t)(2-t)^2 \end{aligned}$$

(b) Let $f_A(t) = 0$, the eigenvalues of A is $\lambda_1 = 4$ and $\lambda_2 = 2$. So

$$A - \lambda_1 I = \begin{pmatrix} -1 & 1 & 1 \\ 2 & 0 & 2 \\ -1 & -1 & 3 \end{pmatrix},$$

solve

$$\begin{cases} -x + y + z = 0 \\ 2x + 2z = 0 \\ -x - y + 3z = 0 \end{cases}$$

then we can get $y = 2x$, $z = -x$, we choose $v_1 = (1, 2, -1)$. Also

$$A - \lambda_2 I = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ -1 & -1 & -1 \end{pmatrix},$$

solve $x + y + z = 0$, then we choose $v_2 = (1, -1, 0)$, $v_2 = (1, 0, -1)$. So the eigenvectors of A is in $\text{span}\{v_1\} \cup \text{span}\{v_2, v_3\}$.

(c) It's easy to check $\mu_A(\lambda_1) = 1 = \gamma_A(\lambda_1)$ and $\mu_A(\lambda_2) = 2 = \gamma_A(\lambda_2)$, so A is diagonalizable. then

$$Q = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix}.$$