MATH2040A/B Linear Algebra II

Midterm Examination 2

Please show all your steps, unless otherwise stated. Answer all **TEN** questions in Part A and Part B. Your submitted solution will be checked carefully to avoid plagiarism. Discussions amongst classmates are strictly prohibited.

Part A. Basic knowledges.

1. (**10pts**) Let
$$\beta = \{2x^2 - x + 1, x^2 + 3x - 2, -x^2 + 2x + 1\}$$
and
$$\beta' = \{9x - 9, x^2 + 21x - 2, 3x^2 + 5x + 2\}$$
be a pair of ordered bases for $P_2(\mathbb{R})$. Find the change of coordinate matrix that changes β'-coordinates into β-coordinates.

2. (**10pts**) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined as the projection onto the line $y = 2x$. See the Figure 1 below for a geometric illustration.

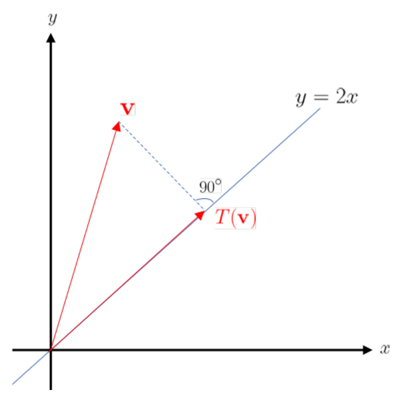

Figure 1: Illustration of Part A Question 2.

Find the matrix representation $[T]_\beta$ of T with respect to the standard ordered basis β of $\mathbb{R}^2$.

3. (**10pts**) For
$$A = \begin{pmatrix} 13 & 1 & 4 \\ 1 & 13 & 4 \\ 4 & 4 & 10 \end{pmatrix} \text{ and } \beta = \left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\},$$
use the definition to find $[L_A]_\beta$. Also, find an invertible matrix Q such that $[L_A]_\beta = Q^{-1}AQ$.

4. (**10pts**) For a vector space $V = P_2(R)$ with an ordered basis $\beta = \{x - x^2, -1 + x^2, -1 - x + x^2\}$, define a linear operator $T \in \mathcal{L}(V)$ by

$$T(a + bx + cx^2) = (-4a + 2b - 2c) - (7a + 3b + 7c)x + (7a + b + 5c)x^2.$$

Compute $[T]_\beta$ and determine whether β is a basis consisting of eigenvectors of T.

5. (**10pts**) Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be a linear transformation defined by:

$$T(a + bx + cx^2) = (4a + 3b + 4c) - (2a + b + 4c)x + 2cx^2$$

Find all the eigenspaces of T. For each eigenspace, find a basis consisting of eigenvectors of T.

Part B. Theory knowledge.

1. (**10pts**) Suppose that U and V are finite-dimensional vector spaces and W is a vector space. Let $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$. Prove that

$$\text{nullity}(ST) \leq \text{nullity}(S) + \text{nullity}(T).$$

2. (**10pts**) Let S and T be linear operators on a finite dimensional vector space V. Prove that $ST = I$ if and only if $TS = I$, where I is the identity linear transformation on V.

3. (**10pts**) Suppose that V and W are finite dimensional vector spaces and that U is a subspace of V. Prove that there exists $T \in \mathcal{L}(V, W)$ such that $N(T) = U$ if and only if

$$\dim(U) \geq \dim(V) - \dim(W).$$

4. (**10pts**) Let $T : V \to W$ be a linear transformation, where V is finite dimensional. Suppose β is a linearly independent subset of V and γ is a basis of $N(T)$.

   (a) Prove that $T(\beta)$ is linearly independent if and only if $\beta \cup \gamma$ is linearly independent.

   (b) Suppose $\beta \cup \gamma$ is linearly independent, deduce that

   $$\text{nullity}(T) \leq \dim(V) - m,$$

   where m is the number of elements in β.

5. (**10pts**) Let T be a linear operator on the vector space V with rank $(T) = k$. Prove that T has at most $k + 1$ distinct eigenvalues.

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