Lecture 4:

Recall: 1. Linearly independent means NOT linearly dependent.

Linearly dependent S,

$\exists$ distinct $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n \in S$ and $\exists\ a_1, a_2, \ldots, a_n \in F$
(not all zero)

such that:

$$a_1\vec{u}_1 + a_2\vec{u}_2 + \cdots + a_n\vec{u}_n = \vec{0}.$$

2. Linearly indepedent S

$\iff$ Each $\vec{x} \in \mathrm{Span}(S)$ can be expressed in a unique way as lin. comb. of S.

$\iff$ $\vec{0} = a_1 \underset{\hat{S}}{\vec{u}_1} + \cdots + a_n \underset{\hat{S}}{\vec{u}_n} \Rightarrow a_1 = a_2 = \cdots = a_n = 0.$

Proposition: Let $S \subset V$ be a subset of a vector space V. Then, the following are equivalent.

(1) S is linearly independent

(2) Each $\vec{x} \in \text{span}(S)$ can be expressed in a unique way as a linear combination of vectors of S.

(3) The only representations of $\vec{0}$ as linear combinations of vectors of S are trivial representations, i.e., if

$$\vec{0} = a_1\vec{u}_1 + \dots + a_n\vec{u}_n \quad \text{for}$$

some $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n \in S$, $a_1, a_2, \dots, a_n \in F$, then we must have $a_1 = a_2 = \dots = a_n = 0$

Example: · For $k = 0, 1, 2, \ldots, n$, let $f_k(x) = 1 + x + x^2 + \cdots + x^k$.

Then: $S = \{ f_0(x), f_1(x), f_2(x), \ldots, f_n(x) \} \subset P_n(F)$ is a linearly independent subset.

$$0 = \vec{0} = a_0 f_0(x) + a_1 f_1(x) + \cdots + a_n f_n(x)$$

$$= a_0 + a_1(1+x) + a_2(1+x+x^2) + \cdots + a_n(1+x+\cdots+x^n)$$

$$= (a_0 + a_1 + \cdots + a_n)1 + (a_1 + a_2 + \cdots + a_n)x$$

$$+ (a_2 + a_3 + \cdots + a_n)x^2 + \cdots + a_n x^n$$

$$\left\{ \begin{array}{l} a_0 + a_1 + \cdots + a_n = 0 \\ \quad a_1 + \cdots + a_n = 0 \\ \qquad a_2 + \cdots + a_n = 0 \\ \qquad\qquad \vdots \\ \qquad\qquad\quad a_n = 0 \end{array} \right. \Rightarrow a_1 = a_2 = \cdots = a_n = 0.$$

Proof: (Sketch of proof)

(1) $\Rightarrow$ (2).: Suppose S is linearly independent.

One scenario: Let $\vec{x} \in \text{Span}(S)$.

Then: $\vec{x} = a_1\vec{u}_1 + \ldots + a_n\vec{u}_n = b_1\vec{u}_1 + \ldots + b_n\vec{u}_n$

$= \cancel{b_1\vec{v}_1 + \ldots + b_m\vec{v}_m}$

Then: $\vec{0} = (a_1 - b_1)\vec{u}_1 + \ldots + (a_n - b_n)\vec{u}_n$

If $a_i - b_i \neq 0$ for some i, then S is linearly dependent.

Contradiction to the fact that S is linearly independent.

$\therefore$ $a_i - b_i = 0$ for all i. $\therefore$ $a_i = b_i$ for all i.

Theorem: Let S be a linearly independent subset of a vector space V. Let $\vec{v} \in V \setminus S$. Then: $S \cup \{\vec{v}\}$ is linearly dependent iff $\vec{v} \in \text{Span}(S)$.

Proof: ($\Rightarrow$) Suppose $S \cup \{\vec{v}\}$ is linearly dependent. Then, we have:

$$a_1 \underbrace{\vec{u}_1}^{=\vec{v}} + a_2\vec{u}_2 + \dots + a_n\vec{u}_n = \vec{0}$$

for some $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n \in S \cup \{\vec{v}\}$ and $a_1, a_2, \dots, a_n \in F \setminus \{0\}$

$\boxed{\because S \text{ is linearly independent, one of } \vec{u}_j\text{'s (say } \vec{u}_1) \text{ must be } \vec{v}.}$

(If not, all $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n \in S$. Then contradiction to the fact that S is lin. ind.)

$$\boxed{\vec{v} = \left(-\frac{a_2}{a_1}\right)\vec{u}_2 + \dots + \left(\frac{-a_n}{a_1}\right)\vec{u}_n} \in \text{Span}(S)$$

($\Leftarrow$) If $\vec{v} \in \mathrm{Span}(S)$, then we can write:

$$\vec{v} = b_1 \vec{v}_1 + \dots + b_m \vec{v}_m$$ for some $\vec{v}_1, \dots, \vec{v}_m \in S$ and $b_1, b_2, \dots, b_m \in F$

$$\Leftrightarrow \overset{\neq 0}{(-1)} \underset{S \cup \{\vec{v}\}}{\vec{v}} + b_1 \underset{S \cup \{\vec{v}\}}{\vec{v}_1} + \dots + b_m \underset{S \cup \{\vec{v}\}}{\vec{v}_m} = \vec{0}$$

(non-trivial linear comb. of elt in $S \cup \{\vec{v}\}$)

$\Downarrow$

$S \cup \{\vec{v}\}$ is linearly dependent.

<u>Definition:</u> A basis for a vector space V is a subset $\beta \subset V$ such that:

- β is linearly independent and
- β spans V, i.e. $\text{Span}(\beta) = V$.

e.g. • F^n : $\{\vec{e}_1 = (1, 0, \ldots, 0), \vec{e}_2 = (0, 1, 0 \ldots, 0), \ldots, \vec{e}_i = (0, \ldots, 0, \overset{i\text{-th}}{1}, 0 \ldots 0), \ldots, \vec{e}_n = (0, 0, \ldots, 1)\}$ is a basis for F^n. (Standard basis)

• $M_{2\times 2}(F)$: $\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & -2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} \right\} \subset M_{2\times 2}(F)$ is a basis for $M_{2\times 2}(F)$

• $\{1, x, x^2, \ldots, x^n\}$ is a basis for $P_n(F)$

• $\{1, x, x^2, \ldots\}$ is a basis for $P(F)$.

- $$\left\{ E_{ij} = \begin{array}{cl} \begin{pmatrix} 0 & & 0 \\ & 1 & \\ 0 & & 0 \end{pmatrix} & \leftarrow i\text{th} \\ \uparrow & \\ j\text{-th} & \end{array} : 1 \le i, j \le n \right\}$$ is a basis for $M_{n \times n}(F)$

$M_{n \times n}(F)$

<u>Proposition</u>: Let V be a vector space and $\beta = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\} \subset V$ is a subset. Then:

if and only if (in) (exist)

β is a basis for V iff $\forall \vec{v} \in V, \exists!\ a_1, a_2, \ldots, a_n \in F$

(for all) (unique)

such that $\vec{v} = a_1\vec{u}_1 + a_2\vec{u}_2 + \ldots + a_n\vec{u}_n$.

Theorem: Let V be a vector space and $\beta = \{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n\} \subset V$.

Then: β is basis for V if and only if: $\forall \vec{v} \in V$, $\exists!$ (unique)

(for all) (in) (there exist)

$a_1, a_2, \dots, a_n \in F$ such that:

$$\vec{v} = a_1\vec{u}_1 + a_2\vec{u}_2 + \dots + a_n\vec{u}_n.$$

V with $\beta = \{$  ,  ,  $\}$

$\in V$

Pineapple is associated with a unique 2, 3, 4 such that

Pineapple = 2  + 3  + 4 

$$\text{Pineapple} \longleftrightarrow \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \in \mathbb{R}^3$$

<u>Proof:</u> ($\Rightarrow$) Suppose β is a basis for V. Let $\vec{v} \in V$.

$\because$ $V = \mathrm{span}(\beta)$

$\therefore$ $\vec{v}$ is a lin. combination of $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n$.

If $\vec{v} = a_1\vec{u}_1 + \dots + a_n\vec{u}_n = b_1\vec{u}_1 + b_2\vec{u}_2 + \dots + b_n\vec{u}_n$

then: $\underset{\substack{\uparrow \\ F}}{(a_1 - b_1)}\vec{u}_1 + \underset{\substack{\uparrow \\ F}}{(a_2 - b_2)}\vec{u}_2 + \dots + \underset{\substack{\uparrow \\ F}}{(a_n - b_n)}\vec{u}_n = \vec{0}$

$\because$ β is linear independent

$\therefore$ $\begin{cases} a_1 - b_1 = 0 \\ a_2 - b_2 = 0 \\ \\ a_n - b_n = 0 \end{cases} \Rightarrow \begin{cases} a_1 = b_1 \\ a_2 = b_2 \\ \vdots \\ a_n = b_n \end{cases} \Rightarrow$ Uniqueness!!

($\Leftarrow$) Suppose $\forall \vec{v} \in V$, $\exists!\ a_1, a_2, \ldots, a_n \in F$ such that:

$$\vec{v} = a_1 \vec{u}_1 + a_2 \vec{u}_2 + \cdots + a_n \vec{u}_n.$$

Then: $V \underset{\supset}{\subset} \mathrm{Span}(\beta) \Rightarrow V = \mathrm{Span}(\beta)$

Also, $\vec{0} = a_1 \vec{u}_1 + a_2 \vec{u}_2 + \cdots + a_n \vec{u}_n$

$\Rightarrow a_1 = a_2 = \cdots = a_n = 0$ by uniqueness.

This implies β is linearly independent.

Remark: Thm is true even for infinite basis.

Lemma: Let S be a linearly dependent subset of a vector space V.

Then: $\exists \vec{v} \in S$ such that $\operatorname{Span}(S \setminus \{\vec{v}\}) = \operatorname{span}(S)$.

Proof: $\because$ S is linearly dependent

$\therefore$ $\vec{0} = b_1 \vec{v} + b_2 \vec{v}_2 + b_3 \vec{v}_3 + \dots + b_m \vec{v}_m$ where

$b_1, b_2, \dots, b_m \in F \setminus \{0\}$ and $\vec{v}, \vec{v}_2, \dots, \vec{v}_m \in S$.

This implies:

$$\vec{v} = -\frac{b_2}{b_1} \vec{v}_2 + \dots + \left(-\frac{b_m}{b_1}\right) \vec{v}_m \in \operatorname{Span}(\{\vec{v}_2, \dots, \vec{v}_m\})$$

$$\therefore \operatorname{Span}(S) = \operatorname{Span}(S \setminus \{\vec{v}\})$$

Now, for any $\vec{w} \in \text{Span}(S)$,

$$\vec{w} = \underset{\overset{\uparrow}{F}}{a_1} \overset{\in S}{\vec{v}} + \underset{\overset{\uparrow}{F}}{a_2} \overset{\in S}{\vec{v}_2} + \dots + \underset{\overset{\uparrow}{F}}{a_n} \overset{\in S}{\vec{v}_n}$$

$$= a_1\left(\left(\frac{b_2}{-b_1}\right)\vec{v}_2 + \dots + \left(\frac{b_m}{-b_1}\right)\vec{v}_m\right) + a_2\vec{v}_2 + \dots + a_n\vec{v}_n \in \text{Span}(S \setminus \{\vec{v}\})$$

$$= \vdots$$

$$\therefore \quad \text{Span}(S) \subseteq \text{Span}(S \setminus \{\vec{v}\})$$

$\supseteq$ Obvious

$$\therefore \quad \text{Span}(S) = \text{Span}(S \setminus \{\vec{v}\})$$