Lecture 19:

Recall:

① $\vec{v} \in \text{Span}(\underbrace{\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}}_{\text{orthogonal}}) \Rightarrow \vec{v} = \sum_{i=1}^{n} \frac{\langle \vec{v}, \vec{v}_i \rangle}{\|\vec{v}_i\|^2} \vec{v}_i$

② G-S process.

$$\overset{S}{\underset{\text{Basis, L.I.}}{\{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_n\}}} \rightsquigarrow \overset{S'}{\underbrace{\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}}_{\text{orthogonal}}}$$

$$\begin{cases} \vec{v}_1 = \vec{w}_1 \\ \vec{v}_k = \vec{w}_k - \sum_{j=1}^{k-1} \frac{\langle \vec{w}_k, \vec{v}_j \rangle}{\|\vec{v}_j\|^2} \vec{v}_j \end{cases} \rightsquigarrow \text{Span}(S') = \text{Span}(S)$$

The above construction of an orthogonal basis is called <u>Gram-Schmidt process.</u>

Example:

Consider $V = P(\mathbb{R})$ equipped with the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(t)\, g(t)\, dt .$$

Let $\beta = \{1, x, x^2, \dots, x^n, \dots\}$ be standard ordered basis for $P(\mathbb{R})$.

Take $\vec{v}_1 = 1$.

Then: $$\vec{v}_2 = x - \frac{\overbrace{\langle x, \vec{v}_1\rangle}^{=0}}{\underbrace{\|\vec{v}_1\|^2}_{=2}} \overbrace{\vec{v}_1}^{=1} = x$$

$$\vec{v}_3 = x^2 - \frac{\overbrace{\langle x^2, \vec{v}_1\rangle}^{=2/3}}{\underbrace{\|\vec{v}_1\|^2}_{2}} \vec{v}_1 - \frac{\overbrace{\langle x^2, \vec{v}_2\rangle}^{=0}}{\|\vec{v}_2\|^2} \vec{v}_2 = x^2 - \frac{1}{3}$$

$$\vec{v}_4 = x^3 - \cancelto{0}{\frac{\langle x^3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}} \vec{v}_1 \boxed{- \frac{\langle x^3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2} - \cancelto{0}{\frac{\langle x^3, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2}} \vec{v}_3$$

$$= x^3 - \frac{3}{5} x \quad \text{and so on, ...}$$

for $P(\mathbb{R})$

This produces an orthogonal basis $\{\vec{v}_1, \vec{v}_2, \ldots\}$, whose elements are called <u>Legendre polynomial</u>.

Corollary: Let V be a non-zero finite-dim inner product space.

Then, V has an orthonormal basis $\beta = \{\vec{v}_1, \ldots, \vec{v}_n\}$ s.t.

$\forall \vec{x} \in V$, we have:
$$\vec{x} = \sum_{i=1}^{n} \langle \vec{x}, \vec{v}_i \rangle \vec{v}_i$$

Corollary: Let V be a non-zero finite-dim inner product space with an orthonormal basis $\beta = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. Let T be a linear operator on V. Let $A = [T]_\beta$. Then: $A_{ij} = \langle T(\vec{v}_j), \vec{v}_i \rangle$.

Proof:
$$[T]_\beta = \begin{pmatrix} \cdots & \begin{matrix} | \\ [T(\vec{v}_j)]_\beta \\ | \end{matrix} & \cdots \end{pmatrix} \qquad T(\vec{v}_j) = \sum_{i=1}^{n} \underbrace{\langle T(\vec{v}_j), \vec{v}_i \rangle}_{= A_{ij}} \vec{v}_i$$

Orthogonal complement

Def: Let S be a non-empty subset of an inner product space V. The orthogonal complement of S is defined as:

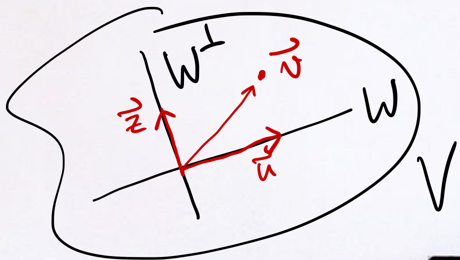
$$S^{\perp} \stackrel{\text{def}}{=} \{ \vec{x} \in V : \langle \vec{x}, \vec{y} \rangle = 0 \text{ for } \forall \vec{y} \in S \}$$

Proposition: Let V be an inner product space and $W \subset V$ a finite-dim subspace of V. Then: $\forall \vec{y} \in V$, $\exists ! \, \vec{u} \in W$ and $\vec{z} \in W^{\perp}$ such that $\vec{y} = \vec{u} + \vec{z}$.

Furthermore, if $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ is an orthonormal basis for W, then:

$$\vec{u} = \sum_{i=1}^{k} \langle \vec{y}, \vec{v}_i \rangle \vec{v}_i$$

The vector $\vec{u} \in W$ is called the orthogonal projection of $\vec{y}$ on W.



$\underline{\text{Proof:}}$ Given $\vec{y} \in V$, we set $\vec{u} \stackrel{\text{def}}{=} \sum_{i=1}^{k} \langle \vec{y}, \vec{v}_i \rangle \vec{v}_i \in W$

and $\vec{z} \stackrel{\text{def}}{=} \vec{y} - \vec{u}$. Then: $\vec{y} = \underset{W}{\underset{\uparrow}{\vec{u}}} + \underset{W^{\perp}\,??}{\underset{\uparrow}{\vec{z}}}$.

Now, $\langle \vec{z}, \vec{v}_j \rangle = \langle \vec{y} - \vec{u}, \vec{v}_j \rangle = \langle \vec{y}, \vec{v}_j \rangle - \langle \vec{u}, \vec{v}_j \rangle$

$$= \langle \vec{y}, \vec{v}_j \rangle - \underbrace{\sum_{i=1}^{k} \langle \vec{y}, \vec{v}_i \rangle \langle \vec{v}_i, \vec{v}_j \rangle}_{\langle \vec{y}, \vec{v}_j \rangle}$$

$$\therefore \langle \vec{z}, \overset{\in W}{\sum_{i=1}^{k} b_k \vec{v}_i} \rangle = \sum_{i=1}^{k} \overline{b_k} \, \cancelto{0}{\langle \vec{z}, \vec{v}_i \rangle} = 0$$

$$= 0$$

$\therefore \quad \vec{z} \in W^{\perp}$

For uniqueness, suppose $\exists\, \vec{u}' \in W$ and $\vec{z}' \in W^\perp$ such that:

$$\vec{y} = \underset{W}{\vec{u}} + \underset{W^\perp}{\vec{z}} = \underset{W}{\vec{u}'} + \underset{W^\perp}{\vec{z}'} \Rightarrow \underset{W}{\vec{u} - \vec{u}'} = \underset{W^\perp}{\vec{z}' - \vec{z}} \in W \cap W^\perp$$

<u>Claim</u>: $W \cap W^\perp = \{\vec{0}\}$

<u>Pf</u>: Take $\vec{w} \in W \cap W^\perp$. Then: $\langle \underset{W}{\vec{w}}, \underset{W^\perp}{\vec{w}} \rangle = 0$

$$\iff \vec{w} = \vec{0}$$

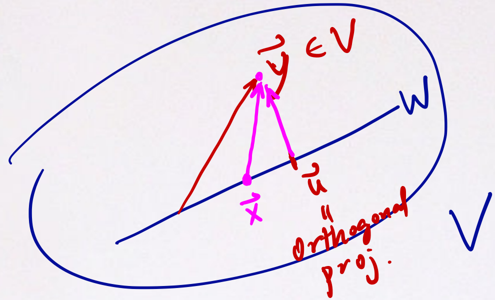
This implies: $\vec{u} - \vec{u}' = \vec{z}' - \vec{z} = \vec{0}$

$\iff \vec{u} = \vec{u}'$ and $\vec{z} = \vec{z}'$.

Corollary: With notations as above, then:

$$\underset{\underset{V}{\uparrow}}{\|\vec{y}} - \underset{\underset{W}{\uparrow}}{\vec{x}}\| \geq \|\underset{\underset{V}{\uparrow}}{\vec{y}} - \underset{\underset{W}{\uparrow}}{\vec{u}}\| \quad \text{for } \forall \vec{x} \in W$$

and equality holds iff $\vec{x} = \vec{u}$



Remark: Orthogonal projection is the vector in W closest to $\vec{y}$.

Proof: Let $\vec{x} \in W$. Then: $\vec{y} = \underset{\substack{\uparrow \\ W}}{\vec{u}} + \underset{\substack{\uparrow \\ W^{\perp}}}{\vec{z}} \Rightarrow \vec{z} = \vec{y} - \vec{u}$

$$\|\vec{y} - \vec{x}\|^2 = \|\vec{u} + \vec{z} - \vec{x}\|^2 = \langle (\underset{\substack{\uparrow \\ W}}{\vec{u}} - \underset{\substack{\uparrow \\ W}}{\vec{x}}) + \underset{\substack{\uparrow \\ W^{\perp}}}{\vec{z}} , (\underset{\substack{\uparrow \\ W}}{\vec{u}} - \underset{\substack{\uparrow \\ W}}{\vec{x}}) + \underset{\substack{\uparrow \\ W^{\perp}}}{\vec{z}} \rangle$$

$$= \langle \vec{u} - \vec{x}, \vec{u} - \vec{x} \rangle + \underset{\substack{\parallel \\ 0}}{\langle \underset{\substack{\uparrow \\ W}}{\vec{u} - \vec{x}}, \underset{\substack{\uparrow \\ W^{\perp}}}{\vec{z}} \rangle} + \underset{\substack{\parallel \\ 0}}{\langle \vec{z}, \vec{u} - \vec{x} \rangle} + \langle \vec{z}, \vec{z} \rangle$$

$$= \|\vec{u} - \vec{x}\|^2 + \|\vec{z}\|^2 \geq \|\vec{z}\|^2 = \|\vec{y} - \vec{u}\|^2$$

The equality holds iff $\|\vec{u} - \vec{x}\|^2 = 0$ iff $\vec{u} = \vec{x}$.

Proposition: Suppose $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is an orthonormal set in an n-dimensional inner product space V. Then:

(a) S can extended to an orthonormal basis $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k, \vec{v}_{k+1}, \ldots, \vec{v}_n\}$ for V.

(b) If $W = \text{span}(S)$, then $S_1 = \{\vec{v}_{k+1}, \ldots, \vec{v}_n\}$ is an orthonormal basis for $W^\perp$

(c) If W is any subspace of V, then:

$$\dim(V) = \dim(W) + \dim(W^\perp)$$

Proof: (a) We first extend S to a basis:

$$\{\overbrace{\underbrace{\vec{v}_1, \ldots, \vec{v}_k}_{\text{L.I.}}}^{S}, \vec{w}_{k+1}, \ldots, \vec{w}_n\} \text{ for } V.$$

Then, we apply the G-S process to this basis.

$\because$ S is orthonormal, $\therefore$ $\vec{v}_1, \ldots, \vec{v}_k$ remains the same during the G-S process.

So, this process gives an orthonormal basis for V of the form

$$\{\underbrace{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k}_{\text{unchanged}}, \underbrace{\vec{v}_{k+1}, \ldots, \vec{v}_n}_{\text{new}}\}$$

(b) Note: • $S_1 = \{\vec{v}_{k+1}, \dots, \vec{v}_n\}$ is linearly independent (∵ it is part of the basis)

• $S_1 \subset W^\perp$ $\quad$ ($W = \text{span}(\{\vec{v}_1, \dots, \vec{v}_k\})$)

$\Rightarrow \text{Span}(S_1) \subset W^\perp$

Suffices to show: $\text{Span}(S_1) = W^\perp$.

For $\vec{x} \in V$, we have $\vec{x} = \sum_{i=1}^{n} \langle \vec{x}, \vec{v}_i \rangle \vec{v}_i$

If $\vec{x} \in W^\perp$, then $\langle \vec{x}, \vec{v}_i \rangle = 0$ for $i = 1, 2, \dots, k$

Then: $\vec{x} = \sum_{i=k+1}^{n} \langle \vec{x}, \vec{v}_i \rangle \vec{v}_i \in \text{Span}(S_1)$

$\therefore W^\perp \subset \text{Span}(S_1) \Rightarrow W^\perp = \text{Span}(S_1)$

(c) For any W, choose an orthonormal basis $\{\vec{v}_1, \ldots, \vec{v}_k\}$ for W and extend it to an orthonormal basis $\{\vec{v}_1, \ldots, \vec{v}_k, \vec{v}_{k+1}, \ldots, \vec{v}_n\}$ for V.

Then:

$$\begin{aligned}\dim(V) = n &= k + (n-k)\\ &= \dim(W) + \dim(W^\perp)\end{aligned}$$