Lecture 16:

Recap:

Def: Given a linear operator T on a vector space V, and a non-zero $\vec{x} \in V$, the subspace

$$W := \operatorname{span}\left(\{T^k(\vec{x}) : k \in \mathbb{N}\}\right) \overset{\text{def}}{=} \operatorname{span}\left(\{\vec{x}, T(\vec{x}), T^2(\vec{x}), \ldots, T^k(\vec{x}), \ldots\}\right)$$

$$\left(T^k \overset{\text{def}}{=} \underbrace{T \circ T \circ \cdots \circ T}_{k \text{ times}} \right)$$

is called T- cyclic subspace of V generated by $\vec{x}$.

Prop: W is the smallest T-invariant subspace of V containing $\vec{x}$.

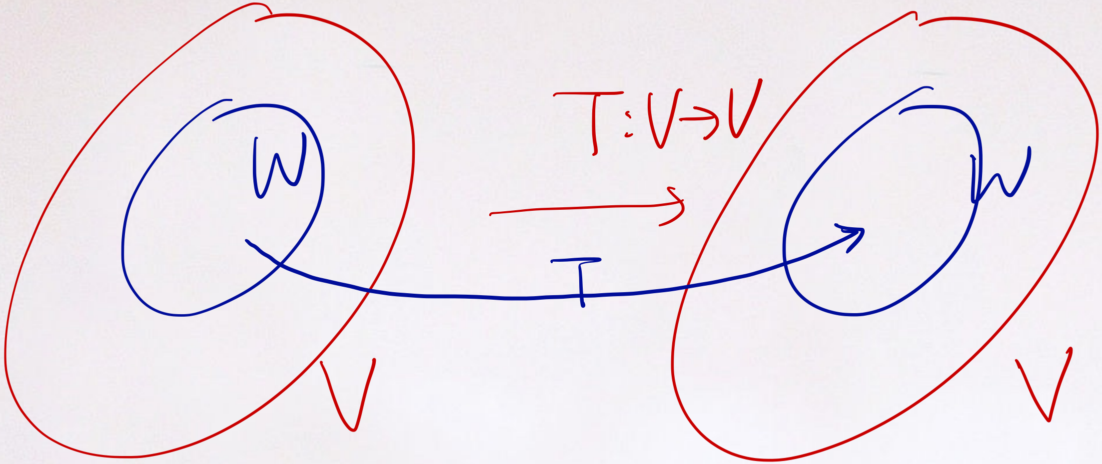
Example: • For $T: P(\mathbb{R}) \to P(\mathbb{R})$ defined by $T(f(x)) = f'(x)$
then T-cyclic subspace generated by x^n is:

$$\text{span}\left(\{x^n, nx^{n-1}, \ldots, n!\,x, n!\}\right) = P_n(\mathbb{R})$$

• Let $T: V \to V$ be linear. Then, a 1-dimensional T-invariant subspace $U \subset V$ is nothing but the span of an eigenvector of T.

[If U = 1-dim T-invariant subspace.
Then, $U = \text{span}\{\vec{v}\}$. Then: $T(\vec{v}) \in U$
$\Rightarrow T(\vec{v}) = \lambda \vec{v}$. $\therefore$ $\vec{v}$ = eigenvector of T.

Also, if $\vec{v} \in V$ is an eigenvector of T, then T-cyclic subspace generated by $\vec{v}$ is also $\text{span}\{\vec{v}\}$. $\left(= \{\vec{v}, \cancel{T(\vec{v})}, \cancel{T^2(\vec{v})} \ldots\}\right)$ $\lambda\vec{v}$ $\lambda^2\vec{v}$



Def: $T|_W : W \to W$ defined by $T|_W(\vec{w}) = T(\vec{w})$

Next time: $f_{T|_W}(t)$ divides $f_T(t)$

Remark: Let $T: V \to V$ be a linear operator on a finite-dim vector space V, and let $W \subset V$ be a T-invariant subspace.

Then, the restriction of T to W, denote it by $T|_W : W \to W$, is well-defined and linear.

Proposition: $f_{T|_W}(t)$ divides $f_T(t)$.

Proof: Choose an ordered basis $\gamma = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ for W and extend it to an ordered basis $\beta = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k, \vec{v}_{k+1}, \dots, \vec{v}_n\}$ for V. Then:

$$[T]_\beta = \begin{pmatrix} | & | & \\ \overset{\in W}{[T(\vec{v}_1)]_\beta} & [T(\vec{v}_2)]_\beta & \cdots \\ | & | & \end{pmatrix} = \begin{array}{c} \begin{array}{cc} \overbrace{\qquad\qquad}^{k} & \qquad \end{array} \\ \left(\begin{array}{cc} \boxed{[T|_W]_\gamma} & \boxed{B} \\ O & \boxed{C} \end{array}\right) \end{array} \begin{array}{l} \big]\, k \\ \\ \end{array}$$

$$f_T(t) = \det\left[\begin{pmatrix} [T_W]_\gamma & B \\ O & C \end{pmatrix} - t\,I\right]$$

$$= \det\begin{pmatrix} [T_W]_\gamma - t\,I_k & B \\ O & C - t\,I_{n-k} \end{pmatrix}$$

$$= \det\left([T_W]_\gamma - t\,I_k\right) \underbrace{\det\left(C - t\,I_{n-k}\right)}_{g(t)}$$

$$= f_{T_W}(t)\; g(t)$$

$\therefore$ $f_{T_W}(t)$ divides $f_T(t)$

Theorem: Let $T: V \to V$ be a linear operator on a finite-dim vector space V and let $W \subset V$ be T-cyclic subspace of V generated by $\vec{v} \neq \vec{0} \in V$. $(W = \text{span}\{\vec{v}, T(\vec{v}), T^2(\vec{v}), \dots\})$

Let $k = \dim(W)$. Then:

(a) $\{\vec{v}, T(\vec{v}), T^2(\vec{v}), \dots, T^{k-1}(\vec{v})\}$ is a basis for W

(b) If $a_0\vec{v} + a_1 T(\vec{v}) + a_2 T^2(\vec{v}) + \dots + a_{k-1}T^{k-1}(\vec{v}) + T^k(\vec{v}) = \vec{0}$,

then the characteristic polynomial of $T|_W$ is:

$$f_{T_W}(t) = (-1)^k(a_0 + a_1 t + a_2 t^2 + \dots + a_{k-1}t^{k-1})$$

Proof: (a) Since $\vec{v} \neq \vec{0}$, then $\{\vec{v}\}$ is linearly independent.

Let j be the largest +ve integer s.t.

$$\beta = \{\vec{v}, T(\vec{v}), \dots, T^{j-1}(\vec{v})\} \text{ is linearly independent.}$$

Such j exists because V is finite-dim.

Let $Z = \text{span}(\beta)$. $\therefore$ $Z \subset W$

Then, $\underbrace{\beta}_{L.I.} \cup T^j(\vec{v})$ is linearly dependent. $\therefore$ $T^j(\vec{v}) \in \overset{Z}{\overset{\shortparallel}{\text{span}(\beta)}}$

$\therefore$ $T^j(\vec{v}) \in Z$.

Now, let $\vec{w} \in Z$. Then $\exists\, b_0, b_1, \dots, b_{j-1} \in F$ s.t.

$$\vec{w} = b_0 \vec{v} + b_1 T(\vec{v}) + \dots + b_{j-1} T^{j-1}(\vec{v})$$

$$T(\vec{w}) = b_0 \underbrace{T(\vec{v})}_{\in Z} + b_1 \underbrace{T^2(\vec{v})}_{\in Z} + \dots + b_{j-2} \underbrace{T^{j-1}(\vec{v})}_{\in Z} + b_{j-1} \underbrace{T^j(\vec{v})}_{\in Z} \in Z$$

$\therefore$ If $\vec{w} \in Z$, then $T(\vec{w}) \in Z$.

$\therefore$ Z is T-invariant subspace containing $\vec{v}$.

$\therefore$ $\underbrace{W}_{\substack{\shortparallel \\ \text{T-cyclic subspace containing } \vec{v}}} \subset Z$. ($\because$ W is smallest T-invariant subspace containing $\vec{v}$)

$\therefore$ $W = Z = \text{span}(\overbrace{\beta}^{\text{L.I.}})$

$\therefore$ β is a basis of W and $j = k$.