Lecture 10:

Recall: • Coordinate representation of $\vec{v} \in V$ w.r.t. $\beta = \{\vec{v}_1, \dots, \vec{v}_n\}$ (ordered)

Write $\vec{v} = \underset{\in F}{a_1}\vec{v}_1 + \underset{\in F}{a_2}\vec{v}_2 + \dots + \underset{\in F}{a_n}\vec{v}_n$

$$[\vec{v}]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in F^n$$

• Matrix representation of a $T: V \to W$, $\beta = \{\vec{v}_1, \dots, \vec{v}_n\}$, $\gamma = \{\vec{w}_1, \dots, \vec{w}_m\}$

$$[T]_\beta^\gamma = \overbrace{\begin{pmatrix} | & & | \\ [T(\vec{v}_1)]_\gamma & \cdots & [T(\vec{v}_n)]_\gamma \\ | & & | \end{pmatrix}}^{n} \Big\} m \in M_{m \times n}$$

Invertibility and Isomorphism

Definition: Let V and W be vector spaces and let $T: V \to W$ be linear. We can say T is invertible if it is bijective (1-1 and onto) so that $\exists\, T^{-1}: W \to V$ such that:

$$T^{-1} \circ T = I_V \quad \text{and} \quad T \cdot T^{-1} = I_W$$

Remark: If V and W are of equal finite-dimensions, then $T: V \to W$ is invertible iff $\operatorname{rank}(T) = \dim(V)$.

$$\operatorname{rank}(T) = \dim(R(T)) \qquad \dim(V) = \dim(W)$$

$$\underbrace{\dim(R(T)) \qquad \dim(W)}_{T \text{ is onto}}$$

Proposition: The inverse $T^{-1}: W \to V$ of an invertible linear transformation $T: V \to W$ is linear.

Proof: Let $\vec{y}_1, \vec{y}_2 \in W$ and $c \in F$.

$\because$ T is bijective $\therefore \exists!\ \vec{x}_1 \in V$ and $\vec{x}_2 \in V$ such that

$$T(\vec{x}_1) = \vec{y}_1 \quad \text{and} \quad T(\vec{x}_2) = \vec{y}_2$$

So,
$$\begin{aligned} T^{-1}(c\vec{y}_1 + \vec{y}_2) &= T^{-1}\left(c\,T(\vec{x}_1) + T(\vec{x}_2)\right) \\ &= T^{-1}\left(T(c\vec{x}_1 + \vec{x}_2)\right) \\ &= c\vec{x}_1 + \vec{x}_2 \\ &= c\,T^{-1}(\vec{y}_1) + T^{-1}(\vec{y}_2) \end{aligned}$$

$\therefore$ T^{-1} is linear.

<u>Example</u>: 1. Let $A \in M_{n\times n}(F)$ is invertible.

Then: $L_A : F^n \to F^n$ defined by $L_A(\vec{x}) = A\vec{x}$.

is invertible and the inverse of L_A is:

$$(L_A)^{-1} = L_{A^{-1}}$$

2. If $T: V \to W$ and $U: W \to Z$ are invertible linear transformations, then: $U \circ T$ is also invertible and

$$(U \circ T)^{-1} = T^{-1} U^{-1}$$

$$\underbrace{\left(T^{-1} \underbrace{\cancel{U^{-1} \circ U}} \circ T \right)}_{Id}$$

Lemma: Suppose $T: V \to W$ is invertible.

Then: $\dim(V) < +\infty$ iff $\dim(W) < +\infty$

And in this case, $\dim(V) = \dim(W)$

Proof: Suppose $\dim(V) = n < +\infty$ and $\beta = \{\vec{x}_1, \dots, \vec{x}_n\}$ is a basis for V. Then: $W = R(T) = \operatorname{span}\{T(\beta)\}$

$$= \operatorname{span}\underbrace{\{T(\vec{x}_1), \dots, T(\vec{x}_n)\}}_{n \text{ elements}}$$

$\therefore \dim(W) \leq n = \dim(V) < +\infty$

Apply the same argument to T^{-1} to show that

$$\dim(V) \leq \dim(W)$$

In particular, if $\dim(V) < +\infty$ and $\dim(W) < +\infty$

then: $\dim(V) \leq \dim(W)$ and $\dim(W) \leq \dim(V) \iff \dim(V) = \dim(W)$

Remark: If $T : V \to W$ is onto,
(linear)

then: $\dim(W) \leq \dim(V)$

Proposition: Let V and W be finite-dimensional vector spaces with ordered basis β and γ respectively.

Let $T: V \to W$ be linear transformation.

Then: T is invertible iff $[T]_\beta^\gamma$ is invertible.

Furthermore,

$$\underbrace{[T^{-1}]_\gamma^\beta}_{\text{Matrix representation of } T^{-1}} = \underbrace{\left([T]_\beta^\gamma\right)^{-1}}_{\text{Inverse of matrix.}}$$

Proof: Suppose T is invertible. Then: $\dim(V) = \dim(W) = n$

Since $T \circ T^{-1} = I_W$, $I_n = [I_W]_\gamma = [T \circ T^{-1}]_\gamma$

$$I_n = [T]_\beta^\gamma \, [T^{-1}]_\gamma^\beta$$

$$\underset{\gamma}{W} \xrightarrow{T^{-1}} \underset{\beta}{V} \xrightarrow{T} \underset{\gamma}{W}$$

Similarly, $T^{-1} \circ T = I_V$. $I_n = [I_V]_\beta = [T^{-1} \circ T]_\beta$

$$I_n = [T^{-1}]_\gamma^\beta \, [T]_\beta^\gamma$$

$\therefore$ $[T]_\beta^\gamma$ is invertible and $\left([T]_\beta^\gamma\right)^{-1} = [T^{-1}]_\gamma^\beta$.

Conversely, suppose $A := [T]_\beta^\gamma$ is invertible. $(\Rightarrow \dim(V) = \dim(W))$

$\because \dim(V) = \dim(W)$

$\therefore$ We only need to show T is one-to-one.

So, suppose $T(\vec{x}_1) = T(\vec{x}_2)$

$$\Leftrightarrow [T(\vec{x}_1)]_\gamma = [T(\vec{x}_2)]_\gamma$$

$$\Rightarrow \underbrace{[T]_\beta^\gamma}_{= A} [\vec{x}_1]_\beta = \underbrace{[T]_\beta^\gamma}_{= A} [\vec{x}_2]_\beta$$

$$\Rightarrow [\vec{x}_1]_\beta = [\vec{x}_2]_\beta \Rightarrow \vec{x}_1 = \vec{x}_2$$

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Corollary: Let V be a finite-dimensional vector space with ordered basis β. Let $T: V \to V$ be a linear transformation.

Then: T is invertible iff $[T]_\beta$ is invertible

Furthermore, $[T^{-1}]_\beta = ([T]_\beta)^{-1}$.

$[L_A]_\beta$ ← standard ordered basis

Corollary: Let $A \in M_{n \times n}(F)$. Then: A is invertible iff L_A is invertible. $(L_A)^{-1} = L_{A^{-1}}$