MATH2040A/B Homework 5 Solution

(Sec 2.1 Q28) Ans: $T(0) = 0 \in \{0\}$ so $\{0\}$ is T invariant. Since $T : V \to V$, V is T invariant. Let $x \in R(T) \subseteq V$. $T(x) \in R(T)$. Finally, Let $x \in N(T)$, $T(T(x)) = T(0) = 0$, so $T(x) \in N(T)$.

(Q29) For $\forall c \in F$ and $\forall x, y \in W$, we have that $cx + y \in W$ and $T_W(cx + y) = T(cx + y) = cT(x) + T(y) = cT_W(x) + T_W(y)$.
Thus T_W is lnear.

(Sec 2.2 Q2) (c) $\begin{pmatrix} 2 & 1 & -3 \end{pmatrix}$

(f) $[T]_\beta^\gamma$ is an $n \times n$ matrix with (i, j)-th entry being 1 if $i + j = n + 1$ and 0 otherwise.

$$[T]_\beta^\gamma = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \iddots & \iddots & 0 \\ 0 & \iddots & \iddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}.$$

(Q5) (a) We have

$$T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$
$$T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence

$$[T]_\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(b) We have

$$T(1) = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \quad T(x) = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}, \quad T(x^2) = \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}.$$

Hence

$$[T]_\beta^\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

(c) We have

$$T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1, \quad T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0, \quad T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 0, \quad T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 1.$$

Hence

$$[T]_\alpha^\gamma = \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}$$

(d) We have $T(1) = 1$, $T(x) = 2$, $T(x^2) = 4$. Therefore

$$[T]_\beta^\gamma = \begin{pmatrix} 1 & 2 & 4 \end{pmatrix}$$

(e) We have

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + (-2)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 4\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence $[A]_\alpha = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 4 \end{pmatrix}$.

(f) $[f(x)]_\beta = \begin{pmatrix} 3 \\ -6 \\ 1 \end{pmatrix}$

(g) $[a]_\gamma = \begin{pmatrix} a \end{pmatrix}$

(Q8) For $\forall x, y \in V$ and $\forall c \in F$, we assume that $[x]_\beta = (x_1, .., x_n)^T, [y]_\beta = (y_1, .., y_n)^T, \beta = \{b_1, .., b_n\}$.
Then $cx + y = c\sum_{i=1}^n x_i b_i + \sum_{i=1}^n y_i b_i = \sum_{i=1}^n (cx_i + y_i) b_i$.
Thus $T(cx + y) = [cx + y]_\beta = (cx_1 + y_1, ..., cx_n + y_n)^T = c(x_1, .., x_n)^T + (y_1, .., y_n)^T = c[x]_\beta + [y]_\beta = cT(x) + T(y)$, which implies T is linear.

(Q11) Let $\{b_1, .., b_k\}$ be a basis of W, we extend it to a basis of V denoted by $\beta = \{b_1, .., b_k, b_{k+1}, ..., b_n\}$.
We will show that this β satisfies the conclusion.
If not, there exist a element of $[T]_\beta : a_{i,j} \neq 0, k + 1 \leq i \leq n, 1 \leq j \leq k$.
Then we have $T(b_j) = \sum_{r=1}^n a_{r,j} b_r$. However, we also get $T(b_j) = \sum_{r=1}^k c_r b_r = \sum_{r=1}^k c_r b_r + \sum_{r=k+1}^n 0 \cdot b_r$ by $\{b_1, .., b_k\}$ be a basis of W and $T(b_j) \in W$.
Since β is a basis of V, the presentation of $T(b_j)$ by β is unique. Thus $a_{i,j} = 0$. Contradiction!

(Q13) Suppose a, b are scalars such that $aT + bU = T_0$ the zero transformation. Since T, U are nonzero, there exists $x, y \in V$ such that $T(x) \neq \overrightarrow{0}$ and $U(y) \neq \overrightarrow{0}$.
If $U(x) \neq \overrightarrow{0}$, then $(aT + bU)(x) = \overrightarrow{0}$. Hence $T(ax) = U(-bx)$. By $\mathsf{R}(T) \cap \mathsf{R}(U) = \{\overrightarrow{0}\}$, we have $aT(x) = T(ax) = -bU(x) = U(-bx) = \overrightarrow{0}$. Since $T(x), U(x)$ are non-zero, $a = b = 0$ and $\{T, U\}$ is linearly independent.
By symmetry, if $T(y) \neq \overrightarrow{0}$, then $\{T, U\}$ is linearly independent. Therefore we left with the case $U(x) = T(y) = \overrightarrow{0}$. Now we have $\overrightarrow{0} = (aT + bU)(x + y) = aT(x) + bU(y)$. Therefore we have $aT(x) = T(ax) = -bU(y) = U(-by)$. By $\mathsf{R}(T) \cap \mathsf{R}(U) = \{\overrightarrow{0}\}$ again, we have $aT(x) = T(ax) = -bU(y) = U(-by) = \overrightarrow{0}$. Since $T(x), U(y)$ are non-zero, $a = b = 0$ and $\{T, U\}$ is linearly independent.

(Sec 2.3 Q3) (a) Note that $U(1) = (1, 0, 1)$, $U(x) = (1, 0, -1)$, $U(x^2) = (0, 1, 0)$. Therefore

$$[U]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

Note that $T(1) = 0+2 = 2$, $T(x) = 3+x+2x = 3+3x$, $T(x^2) = 2x(3+x)+2x^2 = 6x+4x^2$. Therefore

$$[T]_\beta = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{pmatrix}.$$

Note that $(UT)(1) = U(2) = (2, 0, 2)$, $(UT)(x) = U(3 + 3x) = (6, 0, 0)$, $(UT)(x^2) = U(6x + 4x^2) = (6, 4, -6)$. Therefore

$$[UT]_\beta^\gamma = \begin{pmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{pmatrix}.$$

We verify that

$$[U]_\beta^\gamma [T]_\beta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{pmatrix} = [UT]_\beta^\gamma.$$

(b)

$$[h(x)]_\beta = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}.$$

$U(h(x)) = (1, 1, 5)$ and $[U(h(x))]_\gamma = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$. We verify that

$$[U]_\beta^\gamma [h(x)]_\beta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} = [U(h(x))]_\gamma.$$

(Q9) Let $T(a_1, a_2) = (a_1, 0), U(a_1, a_2) = (a_2, a_2)$. It is easy to check that T and U are linear. Then we have $UT(a_1, a_2) = U(T(a_1, a_2)) = U(a_1, 0) = (0, 0), \forall (a_1, a_2) \in F^2$ and $TU(0, 1) = T(U(0, 1)) = T(1, 1) = (1, 0) \neq (0, 0)$.
Let $\beta = \{(1, 0), (0, 1)\}, A = [U]_\beta, B = [T]_\beta$.

(Q11)

$$\begin{aligned} T^2 = T_0 &\iff T(T(u)) = 0, \forall u \in V \\ &\iff T(u) \in N(T), \forall u \in V \\ &\iff R(T) \subseteq N(T) \end{aligned}$$