MATH2040A/B Homework 2 Solution

(Sec 2.1 Q2) Ans:

$$T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Hence T is obviously linear. Let $(a_1, a_2, a_3) \in N(T)$ reduces to $a_1 = a_2, a_3 = 0$. So $N(T) = \{(a_1, a_1, 0) \in \mathbb{R}^3 | a_1 \in \mathbb{R}\}$. Fix $a_1' \in \mathbb{R}$, $0 \neq a_1' \in \mathbb{R}$ we see that the basis is $\{(a_1', a_1', 0)\}$ hence $N(T)$ is of dimension 1. By Theorem 2.2, $R(T) = span(\{T(0,1,0), T(0,0,1), T(1,1,0)\}) = span(\{(-1,0),(0,2)\})$. $\{(-1,0),(0,2)\}$ is linearly independent so it is a basis for $R(T)$ and hence $R(T)$ is of dimension 2. Finally, $\dim R(T) + \dim N(T) = 2+1 = \dim \mathbb{R}^3$ and it is onto.

(Sec 2.1 Q4) Ans:

$$\begin{aligned} T \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} &= \begin{bmatrix} 2a_{11} - a_{12} & a_{13} + 2a_{12} \\ 0 & 0 \end{bmatrix} \\ &= 2 \begin{bmatrix} a_{11} & 0 \\ 0 & 0 \end{bmatrix} + (-1) \begin{bmatrix} a_{12} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & a_{13} \\ 0 & 0 \end{bmatrix} + 2 \begin{bmatrix} 0 & a_{12} \\ 0 & 0 \end{bmatrix} \end{aligned}$$

Is the addition of four linear transformation and hence T is linear. Let $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \in N(T)$ reduces to $a_{12} = 2a_{11}$, $a_{13} = -4a_{11}$. Hence a basis for $N(T)$ is

$$\{\begin{bmatrix} 1 & 2 & -4 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\}$$

and hence $N(T)$ is of dimension 4.
By Theorem 2.2,

$$\begin{aligned} R(T) &= span(T(\{\begin{bmatrix} 1 & 2 & -4 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\}) \\ &= span\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\} \end{aligned}$$

Hence $R(T)$ is of dimension 2.
Finally, $\dim R(T) + \dim N(T) = 2 + 4 = \dim M_{2\times 3}$ and it is neither onto nor one to one.

(Sec 2.1 Q12) Ans: If there is, then $(2,1) = T(-2,0,-6) = T(-2(1,0,3)) = -2(1,1) = (-2,-2)$, which is impossible.

(Sec 2.1 Q13) If $\sum_{i=0}^{k} a_i v_i = 0$, then we have $T(\sum_{i=0}^{k} a_i v_i = 0) = \sum_{i=0}^{k} a_i T(v_i) = \sum_{i=0}^{k} a_i w_i = 0 = 0$ and this implies $a_i = 0$ for any i, which means $\{v_1, \cdots, v_k\}$ is linearly independent.

(Sec 2.1 Q14) Ans:

(a) ($\Rightarrow$): T is one to one, then $N(T) = 0$. Let $\{v_1, ..., v_k\}$ be a linearly independent subset in V, consider

$$\begin{aligned} a_1 T(v_1) + ... + a_k T(v_k) &= 0 \\ T(a_1 v_1 + ... + a_k v_k) &= 0 \end{aligned}$$

$a_1 v_1 + ... + a_k v_k \in N(T) \implies a_1 v_1 + ... + a_k v_k = 0 \implies a_1, ..., a_k = 0$.
($\Leftarrow$): Let $\{v_1, ..., v_n\}$ be a basis for V. Then $\{T(v_1), ..., T(v_n)\}$ is a basis for $R(T)$. By dimension formula, $N(T) = 0$, hence T is one to one.

(b) ($\Rightarrow$): Consider

$$\begin{aligned} a_1T(s_1) + ... + a_nT(s_n) &= 0 \\ T(a_1s_1 + ... + a_ns_n) &= 0, \end{aligned}$$

where $s_1, ..., s_n \in S$.
$a_1s_1 + ... + a_ns_n \in N(T) \implies a_1s_1 + ... + a_ns_n = 0 \implies a_1, ..., a_n = 0$.
Hence $T(S)$ is linearly independent.
($\Leftarrow$): Let $s_1, ..., s_n \in S$. Consider

$$\begin{aligned} a_1s_1 + ... + a_ns_n &= 0 \\ T(a_1s_1 + ... + a_ns_n) &= T(0) = 0 \\ a_1T(s_1) + ... + a_nT(s_n) &= 0 \end{aligned}$$

Hence $a_1, ..., a_n = 0$ since $T(S)$ linearly independent.

(c) By Theorem 2.2 and T is onto, $W = R(T) = span(\{T(v_1), ..., T(v_n)\})$. By (b), $\{T(v_1), ..., T(v_n)\}$ is linearly independent. Hence $\{T(v_1), ..., T(v_n)\}$ is a basis for W.

(Sec 2.1 Q15) Ans:

$$T((cf + g)(x)) = \int_0^x (cf + g)(t)dt = c\int_0^x f(t)dt + \int_0^x g(t)dt = cT(f(x)) + T(g(x)).$$

Hence T is linear.
Let $f \in N(T)$, $f = \sum_{i=0}^n a_ix^i$. $T(f(x)) = 0$ if and only if $\sum_{i=0}^n a_ii + 1x^{i+1} = 0$. By linearly independency of $\{x, ..., x^{n+1}\}$ we get $a_0 = ... = a_n = 0$.
Since if $f = \sum_{i=0}^n a_ix^i$, $T(f(x)) = \sum_{i=0}^n a_ii + 1x^{i+1}$, which fails to map functions to the constant functions so T is not onto.

(Sec 2.1 Q16) Ans: Differentiation is linear from high school calculus.
Obviously $T(x + 1) = T(x + 2)$ so T fails to be one to one.
Given $g = \sum_{i=0}^m g_ix^i$, then $T(\sum_{i=0}^m g_ii + 1x^{i+1}) = g$ so T is one to one.

(Sec 2.1 Q17) Ans :

(a) From dimension formula, $\dim R(T) \leq \dim V < \dim W$, so T cannot be onto.

(b) From dimension formula, $\dim N(T) = \dim V - \dim R(T) \geq \dim V - \dim W > 0$, so T cannot be one to one.

(Sec 2.1 Q19) Ans: Simply consider $T : \mathbb{R} \to \mathbb{R}, T(x) = x$ and $U : \mathbb{R} \to \mathbb{R}, U(x) = 2x$.

(Sec 2.1 Q20) To prove $A = T(V_1)$ is a subspace, first we can check $T(0) = 0 \in A$ by T is linear. Then for any $y_1, y_2 \in A$, we have some $x_1, x_2 \in V$ such that $T(x_1) = y_1$ and $T(x_2) = y_2$, hence we have $T(x_1 + x_2) = y_1 + y_2 \in A$. Simlilarly, we can also check $ay_1\ inA$ and then is a subspace.

To prove $B = \{x \in V : T(x) \in W\}$ is a subspace, firstly we know $T(0) = 0 \in W_1$ and so $0 \in B$. For any $x_1, x_2 \in B$, we have that $T(x_1), T(x_2) \in W_1$ so $T(x_1 + x_2) = T(x_1) + T(x_2) \in W_1$ and $T(cx_1) = cT(x_1) \in W_1$, which means $x_1 + x_2$ and cx_1 are in B, then B is a subspace.