MATH2040A/B Homework 2 Solution

(Sec 1.4 Q5) Ans: (e) True (g) True

(Sec 1.4 Q6) Ans: $\forall(x_1, x_2, x_3) \in \mathbb{F}^3$, we may assume

$$y_1(1,1,0) + y_2(1,0,1) + y_3(0,1,1) = (x_1, x_2, x_3)$$

and solve the system of linear equation. We got

$$\begin{aligned} y_1 &= \frac{1}{2}(x_1 - x_2 + x_3), \\ y_2 &= \frac{1}{2}(x_1 + x_2 - x_3), \\ y_3 &= \frac{1}{2}(-x_1 + x_2 + x_3) \end{aligned}$$

(Sec 1.4 Q12) Ans: To prove it's sufficient we can use Theorem 1.5 and then we know $W = \text{span}(W)$ is a subspace. To prove it's necessary we can also use Theorem 1.5. Since W is a subspace contains W, we have $\text{span}(W) \subset W$. On the other hand, it's natural that $W \subset \text{span}(W)$.

(Sec 1.4 Q13) Ans: $\forall x \in \text{span}(S_1)$, $x = \sum_{i=1}^n a_i e_i$, where $e_1, \cdots, e_n$ are in S_1 and so in S_2, so $x \in \text{span}(S_2)$, which means $\text{span}(S_1) \subset \text{span}(S_2)$. Since $S_2 \subset V$, by Theorem 1.5 we have $\text{span}(S_2) \subset V$. We also know $V = \text{span}(S_1) \subset \text{span}(S_2)$, so $V = \text{span}(S_2)$.

(Sec 1.4 Q15) Ans: $\forall x \in \text{span}(S_1 \cap S_2)$, $x = \sum_{i=1}^n a_i e_i$, where $e_1, \cdots, e_n$ are in $S_1 \cap S_2$, so x is in $\text{span}(S_1)$ and also in $\text{span}(S_2)$, which means $x \in \text{span}(S_1) \cap \text{span}(S_2)$. Let $S_1 = \{(0,1)\}$, $S_2 = \{(1,0)\}$, so $\text{span}(S_1 \cap S_2) = \{(0,0)\} = \text{span}(S_1) \cap \text{span}(S_2)$. Let $S_1 = \{(0,1),(1,0)\}$, $S_2 = \{(-1,0),(0,-1)\}$, so $\text{span}(S_1 \cap S_2) = \{(0,0)\} \neq \mathbb{R}^2 = \text{span}(S_1) \cap \text{span}(S_2)$.

(Sec 1.5 Q2) Ans: (d) Linearly dependent. (h) Linearly independent. (j) Linearly dependent.

(Sec 1.5 Q9) Ans: It's sufficient since if $u = tv$ for some $t \in \mathbb{F}$ then we have $u - tv = 0$. While it's also necessary since if $au + bv = 0$ for some $a, b \in \mathbb{F}$ with at least one not zero, then we may assume $a \neq 0$ and then $u = -\frac{b}{a}v$.

(Sec 1.5 Q14) Ans: By the definition it's easy to prove the sufficiency. Now we are going to prove the necessity. If S is linearly dependent, S can be $\{0\}$. Let $S \neq \{0\}$ is linearly dependent, then we have $a_0u_0 + a_1u_1 + \cdots + a_nu_n = 0$, so $v = u_0 = \frac{1}{a_0}(a_1u_1 + \cdots a_nu_n)$.

(Sec 1.5 Q16) Ans: We can prove it by contrapositive statement.

$\Rightarrow$: If there is a finite subset $\{u_1, u_2, \cdots, u_n\} \subset S$ is linearly dependent, then there are some not all zero $a_1, a_2, \cdots, a_n \in \mathbb{R}$ such that

$$a_1u_1 + a_2u_2 + \cdots + a_nu_n = 0,$$

Then S is also linearly dependent.

$\Leftarrow$: If S is linearly dependent, then there exist vectors $u_1, u_2, \cdots, u_n \in S$ and some not all zero $a_1, a_2, \cdots, a_n \in \mathbb{R}$ such that

$$a_1u_1 + a_2u_2 + \cdots + a_nu_n = 0,$$

Then finite subset $\{u_1, u_2, \cdots, u_n\} \subset S$ is also linearly dependent.

(Sec 1.5 Q19) Ans: If there are some scalars $a_1, a_2, \cdots, a_n \in \mathbb{R}$ such that

$$a_1 A_1^t + a_2 A_2^t + \cdots a_n A_n^t = 0,$$

then we have $(a_1 A_1 + a_2 A_2 + \cdots + a_n A_n)^t = 0$, which is equals to $a_1 A_1 + a_2 A_2 + \cdots + a_n A_n = 0$. Since $\{A_1, A_2, \cdots, A_n\}$ is linearly independent, we know that $a_1 = a_2 = \cdots = a_n = 0$ and $\{A_1^t, A_2^t, \cdots, A_n^t\}$ is linearly independent.