THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2040A/B (First Term, 2018-19)
Linear Algebra II
Solution to Homework 3

Sec. 1.6

Q4 Sol: No, they don't generate $P_3(R)$. What we need to do is to find a polynomial in $P_3(R)$ but not in $span\{v_1, v_2, v_3\}$, where $v_1 = x^3 - 2x^2 + 1, v_2 = 4x^2 - x + 3, v_3 = 3x - 2$.

Let $u = v_1 + v_2 + v_3 + 1$, it is obvious that $u \in P_3(R)$. We will show that $u \notin span\{v_1, v_2, v_3\}$.

If $u \in span\{v_1, v_2, v_3\}$, we have that $v_1 + v_2 + v_3 + 1 = av_1 + bv_2 + cv_3$.

Comparing the coefficient of x^3, we have $a = 1$.

Since $a = 1$, minus v_1 at the both side and comparing the coefficient of x^2, we have $b = 1$.

Since $a = 1$ and $b = 1$, minus $v_1 + v_2$ at the both side and comparing the coefficient of x, we have $c = 1$.

Thus we have $v_1 + v_2 + v_3 + 1 = v_1 + v_2 + v_3$, contradiction arises! So $u \notin span\{v_1, v_2, v_3\}$.

Q11 Sol: **Claim 1**: $\{u + v, au\}$ is a basis for V.

Indeed, suppose c_1, c_2 are scalars such that $c_1(u + v) + c_2(au) = \vec{0}$. Then

$$(c_1 + c_2 a)u + c_1 v = \vec{0}$$

and by linear independence of $\{u, v\}$, $c_1 + c_2 a = c_1 = 0$. As $a \neq 0$, on solving, we get $c_1 = c_2 = 0$. It implies that $\{u + v, au\}$ is linearly independent.

Because $\{u, v\}$ is a basis for V, V is of dimension 2 and by Corollary 2 in Sec. 1.6, $\{u + v, au\}$ is a basis for V.

(**Alternatively**, as $V = \text{span}\{u, v\}$, $\forall w \in V$, $\exists$ scalars c_1, c_2 such that

$$w = c_1 u + c_2 v = c_2(u + v) + a^{-1}(c_1 - c_2)(au) \quad (\because a \neq 0).$$

Hence, $V = \text{span}\{u + v, au\}$.

To conclude, since $\{u + v, au\}$ spans V and is linearly independent, it is a basis for V.)

Claim 2: $\{au, bv\}$ is a basis for V.

Indeed, suppose c_1, c_2 are scalars such that $c_1(au) + c_2(bv) = \vec{0}$. By linear independence of $\{u, v\}$, $c_1 a = c_2 b = 0$. As $a \neq 0$ and $b \neq 0$, $c_1 = c_2 = 0$. It implies that $\{au, bv\}$ is linearly independent.

Because $\{u, v\}$ is a basis for V, V is of dimension 2 and by Corollary 2 in Sec. 1.6, $\{au, bv\}$ is a basis for V.

(**Alternatively**, as $V = \text{span}\{u, v\}$, $\forall w \in V$, $\exists$ scalars c_1, c_2 such that

$$w = c_1 u + c_2 v = (a^{-1}c_1)(au) + (b^{-1}c_2)(bv) \quad (\because a \neq 0 \text{ and } b \neq 0).$$

Hence, $V = \text{span}\{au, bv\}$.

To conclude, since $\{au, bv\}$ spans V and is linearly independent, it is a basis for V.)

Q12 Sol: Suppose a, b, c are scalars such that $a(u+v+w)+b(v+w)+cw=\vec{0}$. Then

$$au+(a+b)v+(a+b+c)w=\vec{0}.$$

By linear independence of $\{u, v, w\}$, $a=a+b=a+b+c=0$. Then, $a=b=c=0$. Therefore, $\{u+v+w, v+w, w\}$ is linearly independent.
Because $\{u, v, w\}$ is a basis for V, V is of dimension 3 and by Corollary 2 in Sec. 1.6, $\{u+v+w, v+w, w\}$ is a basis for V.
(**Alternatively**, as $V=\text{span}\{u, v, w\}$, $\forall x \in V$, $\exists$ scalars a, b, c such that

$$x=au+bv+cw=a(u+v+w)+(b-a)(v+w)+(c-b-a)w.$$

Thus, $V=\text{span}\{u+v+w, v+w, w\}$.
Finally, as $\{u+v+w, v+w, w\}$ spans V and is linearly independent, it is a basis for V.)

Q14 Sol: We only need to find the basis for them.
Let $B_1=\{(0,1,0,0,0),(0,0,0,0,1),(1,0,1,0,0),(1,0,0,1,0)\}$, we will show that B_1 is a basis of W_1.
It is obvious that B_1 is linearly independent and B_1 is a subset of W_1. Then we show that B_1 generates W_1.
$\forall v \in W_1$, we denote v by $(a_3+a_4, a_2, a_3, a_4, a_5)$. Then we have $v=a_2(0,1,0,0,0)+a_5(0,0,0,0,1)+a_3(1,0,1,0,0)+a_4(1,0,0,1,0) \in span(B_1)$
So B_1 is a basis of W_1 and the dimension of W_1 is 4.

Let $B_2=\{(0,1,1,1,0),(1,0,0,0,-1)\}$, we will show that B_2 is a basis of W_2.
It is obvious that B_2 is linearly independent and B_2 is a subset of W_2. Then we show that B_1 generates W_2.
$\forall v \in W_2$, we denote v by $(a_1, a_2, a_2, a_2, -a_1)$. Then we have $v=a_2(0,1,1,1,0)+a_1(1,0,0,0,-1) \in span(B_2)$
So B_2 is a basis of W_2 and the dimension of W_2 is 2.

Q15 Sol: $\forall i, j \in \{1, ..., n\}$, denote by E^{ij} the $n \times n$ matrix in which the only nonzero entry is a 1 in the ith tow and jth column.

Method 1: Choose $l \in \{1, ..., n\}$. $\forall i \in \{1, ..., n\}$ with $i \neq l$, define $H^i=E^{ii}-E^{ll}$.
It is clear that the set

$$B=\{E^{ij} : i, j \in \{1, ..., n\}, i \neq j\} \cup \{H^i : i \in \{1, ..., n\}, i \neq l\}$$

is a subset of W. We claim that it is indeed a basis for W.
Suppose $A \in W$. Then $\exists$ scalars a_{ij}'s for $i, j \in \{1, ..., n\}$ such that $A=\sum_{i=1}^n \sum_{j=1}^n a_{ij}E^{ij}$.
As A has trace zero, i.e. $\sum_{i=1}^n a_{ii}=0$, or equivalently, $a_{ll}=-\sum_{i=1, i\neq l}^n a_{ii}$. Then

$$A=\sum_{\substack{1\leq i,j\leq n\\ i\neq j}} a_{ij}E^{ij}+\sum_{k=1, k\neq l}^{n-1} a_{kk}H^k.$$

It implies that B spans W.
Suppose a_{ij}'s for $i, j \in \{1, ..., n\}$ with $i \neq j$ and $b_1, ..., b_{l-1}, b_{l+1}, ..., b_n$ are scalars such that

$$\sum_{\substack{1 \leq i,j \leq n \\ i \neq j}} a_{ij} E^{ij} + \sum_{k=1, k \neq l}^{n} b_k H^k = 0.$$

By comparing entries of matrices on both sides, we have $a_{ij} = 0 \ \forall i, j \in \{1, ..., n\}$ with $i \neq j$, and $b_k = 0 \ \forall k \in \{1, ..., n\}$ with $k \neq l$. Thus, B is linearly independent.
All in all, we see that B is a basis for W. Hence,

$$\dim W = n(n-1) + (n-1) = n^2 - 1.$$

Method 2: $\forall i \in \{1, ..., n-1\}$, define $H^i = E^{ii} - E^{i+1,i+1}$.
It is clear that the set

$$B = \{E^{ij} : i, j \in \{1, ..., n\}, i \neq j\} \cup \{H^i : i \in \{1, ..., n-1\}\}$$

is a subset of W. We claim that it is indeed a basis for W.
Suppose $A \in W$. Then $\exists$ scalars a_{ij}'s for $i, j \in \{1, ..., n\}$ such that $A = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} E^{ij}$.
We want to find scalars $b_1, ..., b_{n-1}$ such that

$$\sum_{i=1}^{n} a_{ii} E^{ii} = \sum_{i=1}^{n-1} b_k H^k = b_1 E^{11} + (b_2 - b_1) E^{22} + \cdots + (b_{n-1} - b_{n-2}) E^{n-1,n-1} - b_{n-1} E^{nn}.$$

As A has trace zero, i.e. $\sum_{i=1}^{n} a_{ii} = 0$, we can solve the above equation to get $b_k = \sum_{i=1}^{k} a_{ii}$ for any $k \in \{1, ..., n-1\}$, whence

$$A = \sum_{\substack{1 \leq i,j \leq n \\ i \neq j}} a_{ij} E^{ij} + \sum_{k=1}^{n-1} \left(\sum_{i=1}^{k} a_{ii} \right) H^k.$$

It implies that B spans W.
Suppose a_{ij}'s for $i, j \in \{1, ..., n\}$ with $i \neq j$ and $b_1, ..., b_{n-1}$ are scalars such that

$$\sum_{\substack{1 \leq i,j \leq n \\ i \neq j}} a_{ij} E^{ij} + \sum_{k=1}^{n-1} b_k H^k = 0.$$

By comparing entries of matrices on both sides, we have $a_{ij} = 0 \ \forall i, j \in \{1, ..., n\}$ with $i \neq j$, and

$$b_1 = b_2 - b_1 = \cdots = b_{n-1} - b_{n-2} = -b_{n-1} = 0,$$

whence $b_k = 0 \ \forall k \in \{1, ..., n-1\}$. Thus, B is linearly independent.
All in all, we see that B is a basis for W. Hence,

$$\dim W = n(n-1) + (n-1) = n^2 - 1.$$

Q20 Sol:

(a)Proof: If $n = 0$, the proof is trivial.

If $n > 0$, S must contain a vector v_1, s.t. v_1 is not the 0 vector. If not, $S = \{0\}$ then S can not generate V. Contradiction arises!

Let $S_1 = v_1$, then we construct S_p from $S_{p-1}, \forall p \geq 2$:

If S is not the subset of $span\{S_{p-1}\}$, we assume $v_p \in S$ but $v_p \notin span\{S_{p-1}\}$. Let $S_p = S_{p-1} \cup \{v_p\}$.

If S is the subset of $span\{S_{p-1}\}$, we stop this scheme.

Claim 1:For the constructed $S_k, \forall k \in Z^+$, S_k is linearly independent.

when $k = 1$, the claim holds since $S_1 = \{v_1\}$ and $v_1 \neq 0$.

We assume the claim holds when $k \leq q - 1$, so when $k = q$, we have $S_q = S_{q-1} \cup \{v_q\}$, S_{q-1} is linearly independent and $v_q \notin span\{S_{q-1}\}$.

So by theorem 1.7, we have S_q is linearly independent. Thus the claim also holds when $k = q$.

Therefore the claim holds for $\forall k \in Z^+$ by induction.

Claim 2: We can not get the set $S_k, k \geq n + 1$.

If the claim 2 doesn't hold, there is a set S_{n+1} such that S_{n+1} is linearly independent and it contains $n + 1$ vector.

So the dimension of $span\{S_{n+1}\}$ is $n+1$. And $span\{S_{n+1}\}$ is the subspace of V because $S_{n+1} \subset S \subset V$. However, the dimension of V is n which is small than the dimension of its subspace. That is impossible. So the claim 2 holds.

From claim 2, we assume that the constructing scheme stops at S_k, i.e. S is the subset of $span\{S_k\}$. So $V = span\{S\} \subset span\{S_k\}$. Thus $V = span\{S_k\}$ since $S_k \subset V$.

Since $V = span\{S_k\}$ and S_k is linearly independent, so S_k is a subset of S that is a basis of V.

(b)Proof: If S only contains k vectors, $k \leq n - 1$.

From (a), we can find a subset of S that is a basis of V. We denote it by U. Then the dimension of V is not larger that $n - 1$ since U is a basis of V and U contains $n - 1$ vector at most. Contradiction arises!

Thus S contains at least n vectors.

Q23 Sol: i. We claim that $v \in W_1$ is a necessary and sufficient conditions such that

$$\dim(W_1) = \dim(W_2).$$

Note that as $\{v_1, ..., v_k\} \subset \{v_1, ..., v_k, v\}$, $W_1 \subset W_2$ (by a lemma in Lecture Note 3).

($\Rightarrow$) Suppose $v \in W_1$. Clearly, $v_i \in \text{span}(\{v_1, ..., v_k\}) = W_1 \ \forall i \in \{1, ..., n\}$. Then $\{v_1, ..., v_k, v\} \subset \text{span}(\{v_1, ..., v_k\})$ and thus (by the same lemma in Lecture Note 3)

$$W_2 = \text{span}\{v_1, ..., v_k, v\} \subset \text{span}(\text{span}(\{v_1, ..., v_k\})) = \text{span}(\{v_1, ..., v_k\}) = W_1$$

Therefore, $W_1 = W_2$ and thus $\dim W_1 = \dim W_2$.

($\Leftarrow$) Suppose $\dim W_1 = \dim W_2$. Because $W_1 \subset W_2$ and $\dim W_1 = W_2$, $W_1 = W_2$ (by a theorem in Lecture Note 3). Thus, $v \in \{v_1, ..., v_k, v\} \subset W_2 = W_1$.

ii. We claim that in case $\dim(W_1) \neq \dim(W_2)$, $\dim(W_2) = \dim(W_1) + 1$.
We first treat the special case when $\{v_1, ..., v_k\}$ is a basis for W_1. Then $\dim(W_1) = k$. Now, it suffices to show that $\{v_1, ..., v_k, v\}$ is a basis for W_2, implying $\dim(W_2) = k + 1$.
By definition, $\{v_1, ..., v_k, v\}$ spans W_2. Suppose $c_0, ..., c_k$ are scalars such that

$$c_0 v + c_1 v_1 + \cdots + c_k v_k = \vec{0}.$$

Then $c_0 v = -\sum_{i=1}^{k} c_i v_i \in W_1$. By the hypothesis that $\dim(W_1) \neq \dim(W_2)$ and (a), $v \notin W_1$. As W_1 is a vector space, $c_0 v \in W_1$ only if $c_0 = 0$, whence $\sum_{i=1}^{n} c_i v_i = \vec{0}$. By linearly independence of $\{v_1, ..., v_k\}$, $c_1 = \cdots = c_k = 0$. Therefore, $\{v_1, ..., v_k, v\}$ is linearly independent. Hence, $\{v_1, ..., v_k, v\}$ is a basis for W_2.
Now we go to the general case when $v_1, ..., v_k$ are arbitrary vectors of W_1. Choose any basis $\{v'_1, ..., v'_{k'}\}$ for W_1. Then $W_1 = \text{span}(\{v'_1, ..., v'_{k'}\})$. We want to show that $W_2 = \text{span}\{v'_1, ..., v'_{k'}, v\}$ so that we can apply the argument in the special case by replacing $v_1, ..., v_k$ by $v'_1, ..., v'_{k'}$.
Since $v_1, ..., v_k \in W_1 = \text{span}\{v'_1, ..., v'_{k'}\}$, $\{v_1, ..., v_k, v\} \subset \text{span}\{v'_1, ..., v'_{k'}, v\}$ and hence $W_2 \subset \text{span}(\{v'_1, ..., v'_{k'}, v\})$. Similarly, because $v'_1, ..., v'_{k'} \in W_1 = \text{span}\{v_1, ..., v_k\}$, $\{v'_1, ..., v'_{k'}, v\} \subset \text{span}\{v_1, ..., v_k, v\}$ and hence $\text{span}(\{v'_1, ..., v'_{k'}, v\}) \subset W_2$. We are done.

Q24 Sol: Let $S = \{f^{(k)}(x)\}, k = 0, 1, ..., n$. We show that S is linearly independent. Since $f(x)$ is a polynomial of degree n, thus $f^{(k)}(x)$ is a polynomial of degree $n - k$.
Assume that $\Sigma_{k=0}^{n} a_k f^{(k)}(x) = 0$, we show that $a_k = 0, , k = 0, 1, ..., n$ thus S is linearly independent.
When $k = 0$, comparing the coefficient of x^n, we have $a_0 = 0$.
We assume that $a_k = 0$ holds when $k \leq p - 1$. The when $k = p$, we have
$\Sigma_{k=p}^{n} a_k f^{(k)}(x) = 0$
Comparing the coefficient of x^{n-p}, we have $a_p = 0$ since the degree of $f^{(k)}(x)$ is $n - p - 1$ at most when $k \geq p + 1$.
Thus $a_k = 0$ also holds when $k = p$.
Therefore a_k must be equal to 0 by induction, $k = 0, 1, ..., n$.
So S is linearly independent and contains $n + 1$ vectors.
Since the dimension of $P_n(R)$ is n+1, we get that S is a basis of $P_n(R)$ by corollary 2(b).
The assertion of Q24 is true by the definition of basis.

Q26 Sol: Let $V = \{f \in \mathsf{P}_n(\mathbb{R}) : f(a) = 0\}$.

Method 1: $\forall i \in \{1, ..., n\}$, define $g_i \in \mathsf{P}_n(\mathbb{R})$ by $g_i(x) = x^i - a^i$. We claim that $\{g_1, ..., g_n\}$ is a basis for V and therefore the dimension of V is n.
First, notice that $\forall i \in \{1, ..., n\}$, $g_i(a) = a^i - a^i = 0$ and hence $g_i \in V$.
Second, suppose $c_1, ..., c_n \in \mathbb{R}$ such that $c_1 g_1 + \cdots c_n g_n = 0$. Then

$$c_n x^n + \cdots c_1 x - (c_n a^n + \cdots + c_1 a) = 0.$$

By comparing coefficients, we get $c_1 = \cdots = c_n = 0$. $\{g_1, ..., g_n\}$ is linearly independent.
Third, fix $f \in V$. $\exists c_0, ..., c_n \in \mathbb{R}$ such that $f(x) = \sum_{i=0}^{n} c_i x^i$. Then $\sum_{i=0}^{n} c_i a^i = 0$, or

equivalently, $c_0 = -\sum_{i=1}^{n} c_i a^i$. We have

$$f(x) = c_n x^n + \cdots + c_1 x + c_0 = c_n g_1(x) + \cdots + c_1 g_1(x).$$

Thus, $\text{span}(\{g_1, ..., g_n\}) = V$.
Eventually, we see that $\{g_1, ..., g_n\}$ is a basis for V and $\dim V = n$.

Method 2: $\forall i \in \{1, ..., n\}$, define $g_i \in \mathsf{P}_n(\mathbb{R})$ by $g_i(x) = (x-a)^i$. We claim that $\{g_1, ..., g_n\}$ is a basis for V and therefore the dimension of V is n.
First, notice that $\forall i \in \{1, ..., n\}$, $g_i(a) = (x-a)^i = 0$ and hence $g_i \in V$.
Second, suppose $c_1, ..., c_n \in \mathbb{R}$ such that $c_1 g_1 + \cdots c_n g_n = 0$. Putting $y = x - a$, the equality becomes $c_n y^n + \cdots + c_1 y = 0$, which yields $c_n = \cdots = c_1$. $\{g_1, ..., g_n\}$ is linearly independent.
Third, fix $f \in V$. Define $h(x) = f(x+a) \in \mathsf{P}_n(\mathbb{R})$. Then $\exists c_0, ..., c_n \in \mathbb{R}$ such that $h(x) = \sum_{i=0}^{n} c_i x^i$. Note that $c_0 = h(0) = f(a) = 0$. Then

$$f(x) = h(x-a) = \sum_{i=1}^{n} c_i g_i(x).$$

Thus, $\text{span}(\{g_1, ..., g_n\}) = V$.
Eventually, we see that $\{g_1, ..., g_n\}$ is a basis for V and $\dim V = n$.

Method 3: $\forall i \in \{1, ..., n\}$, define $g_i \in \mathsf{P}_n(\mathbb{R})$ by $g_i(x) = (x-a)x^{i-1}$. We claim that $\{g_1, ..., g_n\}$ is a basis for V and therefore the dimension of V is n.
First, notice that $\forall i \in \{1, ..., n\}$, $g_i(a) = (a-a)a^{i-1} = 0$ and hence $g_i \in V$.
Second, suppose $c_1, ..., c_n \in \mathbb{R}$ such that $c_1 g_1 + \cdots c_n g_n = 0$. Then

$$c_n x^n + (c_{n-1} - ac_n)x^{n-1} + (c_{n-2} - ac_{n-1})x^{n-2} + \cdots + (c_1 - ac_2)x - ac_1 = 0.$$

So $c_n = c_{n-1} - ac_n = c_{n-2} - ac_{n-1} = \cdots = c_1 - ac_2 = -ac_1 = 0$. On solving, we get $c_1 = \cdots = c_n = 0$. $\{g_1, ..., g_n\}$ is linearly independent.
Third, fix $f \in V$. Since $f(a) = 0$, by Factor Theorem, $\exists g \in \mathsf{P}_{n-1}(\mathbb{R})$ such that $f(x) = (x-a)g(x)$. Then $\exists c_1, ..., c_n \in \mathbb{R}$ such that $g(x) = \sum_{i=1}^{n} c_i x^{i-1}$, whence $f(x) = \sum_{i=1}^{n} c_i g_i(x) \in \text{span}(\{g_1, ..., g_n\})$. Thus, $\text{span}(\{g_1, ..., g_n\}) = V$.
Eventually, we see that $\{g_1, ..., g_n\}$ is a basis for V and $\dim V = n$.

Method 4*: (This is an approach to prove the statement assuming that we have already learnt knowledges in Sec. 2.1 - 2.4.)
Define $T : \mathsf{P}_{n-1}(\mathbb{R}) \to \mathsf{P}_n(\mathbb{R})$ by $g(x) \mapsto (x-a)g(x)$. T is in fact a map from $\mathsf{P}_{n-1}(\mathbb{R})$ to V since $\forall g \in \mathsf{P}_{n-1}(\mathbb{R})$, $T(g)(a) = (a-a)g(a) = 0$. $\forall f, g \in \mathsf{P}_{n-1}(\mathbb{R})$, $\forall c \in \mathbb{R}$,

$$T(f+g)(x) = (x-a)(f+g)(x) = (x-a)f(x) + (x-a)g(x) = T(f)(x) + T(g)(x),$$
$$T(cf)(x) = (x-a)(cf)(x) = c(x-a)f(x) = cT(f)(x).$$

Therefore, $T : \mathsf{P}_{n-1}(\mathbb{R}) \to V$ is a linear transformation.
If $g(x) \in \mathsf{P}_{n-1}(\mathbb{R})$ and $(x-a)g(x) = T(g)(x) = 0$, then clearly $g(x) = 0$. Hence, $\mathsf{N}(T) = \{0\}$. By Theorem 2.4 in Sec. 2.1, T is one-to-one. $\forall f(x) \in V$, by Factor Theorem $\exists g(x) \in \mathsf{P}(\mathbb{R})$ such that $f(x) = (x-a)g(x)$ and by comparing degrees, we know

that $g(x) \in \mathsf{P}_{n-1}(\mathbb{R})$, whence $f = T(g)$. Therefore, T is also onto. As T is one-to-one and onto, it is invertible.
Finally, by a lemma in Sec. 2.4 (see page 101), $\dim V = \dim \mathsf{P}_{n-1}(\mathbb{R}) = n$.

Method 5*: (This is an approach to prove the statement assuming that we have already learnt knowledges in Sec. 2.1, especially properties of null spaces)
Define $T : \mathsf{P}_n(\mathbb{R}) \to \mathbb{R}$ by $f \mapsto f(a)$. Note that $\forall f, g \in \mathsf{P}_n(\mathbb{R})$, $\forall c \in \mathbb{R}$,

$$T(f+g) = (f+g)(a) = f(a) + g(a) = T(f) + T(g),$$
$$T(cf) = cf(a) = cT(f).$$

Hence, T is a linear transformation from $\mathsf{P}_n(\mathbb{R})$ to $\mathbb{R}$. Now we see that the null space $\mathsf{N}(T)$ of T is equal to V. On the other hand, $\forall c \in \mathbb{R}$, $T(f_c) = f_c(a) = c$, where $f_c \in \mathsf{P}_n(\mathbb{R})$ is the constant polynomial with constant term c. Therefore, the range $\mathsf{R}(T)$ of T is equal to $\mathbb{R}$. Now we apply Dimension Theorem (Theorem 2.3 in Sec. 2.1):

$$\text{nullity}(T) + \text{rank}(T) = \dim \mathsf{P}_n(\mathbb{R}),$$
$$\therefore \dim V + \dim \mathbb{R} = \dim \mathsf{P}_n(\mathbb{R}),$$
$$\therefore \dim V = \mathsf{P}_n(\mathbb{R}) - \dim \mathbb{R} = (n+1) - 1 = n.$$

Q31 Sol: (a) As $W_1 \cap W_2 \subset W_2$, $\dim(W_1 \cap W_2) \leq \dim(W_2) = n$ by Theorem 1.11, Sec 1.6.

(b) We assume $S_1 = \{v_1, ..., v_m\}$ is a basis of W_1 and $S_2 = \{u_1, ..., u_n\}$ is a basis of W_2.
It is obvious that $S = S_1 \cup S_2$ generates $W_1 + W_2$.
By theorem 1.9, we get a subset S' of S such that S' is a basis of $W_1 + W_2$.
S' contains $m+n$ vectors at most since $S = S_1 \cup S_2$ contains $m+n$ vectors at most.
Therofore, $\dim(W_1 + W_2) \leq m + n$.