

# MMAT5390: Mathematical Image Processing

## Solutions to Chapter 3 Exercises

1. (a) As  $A = U\Sigma V^T$ ,

$$A^T A = (U\Sigma V^T)^T (U\Sigma V^T) = V\Sigma^T U^T U\Sigma V^T = V\Sigma^T \Sigma V^T$$

and

$$AA^T = (U\Sigma V^T)(U\Sigma V^T)^T = U\Sigma V^T V\Sigma^T U^T = U\Sigma\Sigma^T U^T.$$

Note that

$$\Sigma^T \Sigma = \begin{cases} \left( \begin{array}{cccc|ccc} \sigma_{11}^2 & 0 & \cdots & 0 & & & \\ 0 & \sigma_{22}^2 & \cdots & 0 & & & \\ \vdots & \vdots & \ddots & \vdots & & & \\ 0 & 0 & \cdots & \sigma_{KK}^2 & & & \\ \hline & \mathbf{0}_{(N-M) \times M} & & & & & \\ \sigma_{11}^2 & 0 & \cdots & 0 & & & \\ 0 & \sigma_{22}^2 & \cdots & 0 & & & \\ \vdots & \vdots & \ddots & \vdots & & & \\ 0 & 0 & \cdots & \sigma_{KK}^2 & & & \end{array} \right) & \text{if } M < N \\ \left( \begin{array}{cccc|ccc} \sigma_{11}^2 & 0 & \cdots & 0 & & & \\ 0 & \sigma_{22}^2 & \cdots & 0 & & & \\ \vdots & \vdots & \ddots & \vdots & & & \\ 0 & 0 & \cdots & \sigma_{KK}^2 & & & \\ \hline \sigma_{11}^2 & 0 & \cdots & 0 & & & \\ 0 & \sigma_{22}^2 & \cdots & 0 & & & \\ \vdots & \vdots & \ddots & \vdots & & & \\ 0 & 0 & \cdots & \sigma_{KK}^2 & & & \\ \hline & \mathbf{0}_{(M-N) \times N} & & & & & \\ \sigma_{11}^2 & 0 & \cdots & 0 & & & \\ 0 & \sigma_{22}^2 & \cdots & 0 & & & \\ \vdots & \vdots & \ddots & \vdots & & & \\ 0 & 0 & \cdots & \sigma_{KK}^2 & & & \\ \hline & \mathbf{0}_{(M-N) \times N} & & & & & \end{array} \right) & \text{if } M \geq N \end{cases}$$

and

$$\Sigma \Sigma^T = \begin{cases} \left( \begin{array}{cccc|ccc} \sigma_{11}^2 & 0 & \cdots & 0 & & & \\ 0 & \sigma_{22}^2 & \cdots & 0 & & & \\ \vdots & \vdots & \ddots & \vdots & & & \\ 0 & 0 & \cdots & \sigma_{KK}^2 & & & \\ \hline \sigma_{11}^2 & 0 & \cdots & 0 & & & \\ 0 & \sigma_{22}^2 & \cdots & 0 & & & \\ \vdots & \vdots & \ddots & \vdots & & & \\ 0 & 0 & \cdots & \sigma_{KK}^2 & & & \\ \hline & \mathbf{0}_{(M-N) \times N} & & & & & \\ \sigma_{11}^2 & 0 & \cdots & 0 & & & \\ 0 & \sigma_{22}^2 & \cdots & 0 & & & \\ \vdots & \vdots & \ddots & \vdots & & & \\ 0 & 0 & \cdots & \sigma_{KK}^2 & & & \\ \hline & \mathbf{0}_{(M-N) \times N} & & & & & \end{array} \right) & \text{if } M \leq N \\ \left( \begin{array}{cccc|ccc} \sigma_{11}^2 & 0 & \cdots & 0 & & & \\ 0 & \sigma_{22}^2 & \cdots & 0 & & & \\ \vdots & \vdots & \ddots & \vdots & & & \\ 0 & 0 & \cdots & \sigma_{KK}^2 & & & \\ \hline \sigma_{11}^2 & 0 & \cdots & 0 & & & \\ 0 & \sigma_{22}^2 & \cdots & 0 & & & \\ \vdots & \vdots & \ddots & \vdots & & & \\ 0 & 0 & \cdots & \sigma_{KK}^2 & & & \\ \hline & \mathbf{0}_{(M-N) \times N} & & & & & \\ \sigma_{11}^2 & 0 & \cdots & 0 & & & \\ 0 & \sigma_{22}^2 & \cdots & 0 & & & \\ \vdots & \vdots & \ddots & \vdots & & & \\ 0 & 0 & \cdots & \sigma_{KK}^2 & & & \\ \hline & \mathbf{0}_{(M-N) \times N} & & & & & \end{array} \right) & \text{if } M > N \end{cases}$$

Hence  $(\sigma_{11}, \sigma_{22}, \dots, \sigma_{KK})$  are the square roots of the largest  $K$  eigenvalues of  $A^T A$  (or  $AA^T$ ) in descending order, and thus the  $K$ -tuple is uniquely determined.

- (b) Suppose  $\{\sigma_{ii} : i = 1, 2, \dots, K\}$  are distinct and nonzero. Then each eigenspace of  $A^T A$  and  $AA^T$  corresponding to eigenvalue  $\sigma_{ii}^2$  has dimension 1, which means that there are exactly two unit eigenvectors to be chosen from each eigenspace, each being the negative of the other. Such eigenvectors are precisely the first  $K$  columns of  $U$  and  $V$ . Combined with the fact that  $\sigma_{ii}$  are in descending order, the first  $K$  columns of  $U$  and  $V$  are uniquely determined up to a change of sign.
- (c) Suppose  $M = N$ . Then the argument from (b) follows.
- (d) A counterexample with nondistinct  $\{\sigma_{ii} : i = 1, 2, \dots, K\}$  is given by:

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2 I_2 I_2 = U I_2 U^T,$$

where  $I_2$  and  $U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$  are unitary.

A counterexample with  $\sigma_{KK} = 0$  is given by:

$$(0 \ 0) = (1)(0 \ 0) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (1)(0 \ 0) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

2. Let  $g \in M_{m \times n}(\mathbb{R})$  with SVD  $g = U\Sigma V^T$ .

For any  $c_1, c_2, \dots, c_r \in \mathbb{R}$ ,

$$\begin{aligned} \sum_{i=1}^r c_i \vec{u}_i &= \sum_{i=1}^r \frac{c_i}{\sigma_i} \sigma_i \vec{u}_i (\vec{v}_i^T \vec{v}_i) \\ &= \left( \sum_{i=1}^r \sigma_i \vec{u}_i \vec{v}_i^T \right) \left( \sum_{j=1}^r \frac{c_j}{\sigma_j} \vec{v}_j \right) \text{ since } \vec{v}_i^T \vec{v}_j = \delta(i-j) &= g \left( \sum_{j=1}^r \frac{c_j}{\sigma_j} \vec{v}_j \right). \end{aligned}$$

Let  $\vec{v} \in \mathbb{R}^n$ . Since  $V$  is unitary,  $\text{span}(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n) = \mathbb{R}^n$  and thus there exist  $c_1, c_2, \dots, c_n \in \mathbb{R}$  such that  $\vec{v} = \sum_{j=1}^n c_j \vec{v}_j$ . Then

$$\begin{aligned} g\vec{v} &= \left( \sum_{i=1}^r \sigma_i \vec{u}_i \vec{v}_i^T \right) \sum_{j=1}^n c_j \vec{v}_j \\ &= \sum_{i=1}^r \sum_{j=1}^n c_i \sigma_j \vec{u}_i \vec{v}_i^T \vec{v}_j \\ &= \sum_{i=1}^r \sum_{j=1}^n c_i \sigma_j \delta(i-j) \vec{u}_i \\ &= \sum_{i=1}^r (c_i \sigma_i) \vec{u}_i. \end{aligned}$$

Hence  $\text{range}(g) = \text{span}(\vec{u}_1, \vec{u}_2, \dots, \vec{u}_r)$ .

For any  $c_{r+1}, c_{r+2}, \dots, c_n \in \mathbb{R}$ ,

$$\begin{aligned} g \left( \sum_{i=r+1}^n c_i \vec{v}_i \right) &= \left( \sum_{j=1}^r \sigma_j \vec{u}_j \vec{v}_j^T \right) \left( \sum_{i=r+1}^n c_i \vec{v}_i \right) \\ &= \sum_{j=1}^r \sum_{i=r+1}^n c_i \sigma_j \vec{u}_j \vec{v}_j^T \vec{v}_i \\ &= \sum_{j=1}^r \sum_{i=r+1}^n c_i \sigma_j \delta(i-j) \vec{u}_j = 0. \end{aligned}$$

Let  $\vec{v} = \sum_{i=1}^n c_i \vec{v}_i \in \mathbb{R}^n$  such that  $g\vec{v} = \vec{0}_m$ . Then for any  $i \in [1, r]$ ,

$$\begin{aligned} c_i &= \vec{v}_i^T \vec{v} \\ &= \vec{u}_i^T \vec{u}_i \vec{v}_i^T \vec{v} \\ &= \frac{1}{\sigma_i} \vec{u}_i^T \left( \sum_{j=1}^r \sigma_j \vec{u}_j \vec{v}_j^T \right) \vec{v} \\ &= \frac{1}{\sigma_i} \vec{u}_i^T g\vec{v} = \frac{1}{\sigma_i} \vec{u}_i^T \vec{0}_m = 0. \end{aligned}$$

and thus  $\vec{v} = \sum_{i=r+1}^n c_i \vec{v}_i$ .

Hence  $\text{null}(g) = \text{span}\{\vec{v}_{r+1}, \vec{v}_{r+2}, \dots, \vec{v}_n\}$ .

3. (a)  $ff^T = \begin{pmatrix} 10 & 0 \\ 0 & 20 \end{pmatrix}$ .

For  $\lambda = 20$ :

$$\left[ \begin{array}{cc|c} -10 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right] \sim \left[ \begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right],$$

which gives unit eigenvector  $\vec{u}_1 = (0, 1)^T$ .

For  $\lambda = 10$ :

$$\left[ \begin{array}{cc|c} 0 & 0 & 0 \\ 0 & -10 & 0 \end{array} \right] \sim \left[ \begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right],$$

which gives unit eigenvector  $\vec{u}_2 = (1, 0)^T$ .

$$\text{Then } \vec{v}_1 = \frac{f^T \vec{u}_1}{\sigma_1} = \frac{1}{2\sqrt{5}} \begin{pmatrix} 1 & 0 \\ 0 & 4 \\ 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{5}}(0, 2, 0, 1)^T, \text{ and}$$

$$\vec{v}_2 = \frac{f^T \vec{u}_2}{\sigma_2} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & 0 \\ 0 & 4 \\ 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{10}}(1, 0, 3, 0)^T.$$

$$f^T f = \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 16 & 0 & 8 \\ 3 & 0 & 9 & 0 \\ 0 & 8 & 0 & 4 \end{pmatrix}.$$

For  $A^T A \vec{v} = 0$ ,

$$\left[ \begin{array}{cccc|c} 1 & 0 & 3 & 0 & 0 \\ 0 & 16 & 0 & 8 & 0 \\ 3 & 0 & 9 & 0 & 0 \\ 0 & 8 & 0 & 4 & 0 \end{array} \right] \sim \left[ \begin{array}{cccc|c} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right],$$

which gives orthonormal eigenvectors  $\vec{v}_3 = \frac{1}{\sqrt{10}}(-3, 0, 1, 0)^T$  and  $\vec{v}_4 = \frac{1}{\sqrt{5}}(0, 1, 0, -2)^T$ .

Hence an SVD of  $A$  is  $A = U\Sigma V^T$ , where

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \Sigma = \begin{pmatrix} 2\sqrt{5} & 0 & 0 & 0 \\ 0 & \sqrt{10} & 0 & 0 \end{pmatrix} \text{ and } V = \frac{1}{\sqrt{10}} \begin{pmatrix} 0 & 1 & -3 & 0 \\ 2\sqrt{2} & 0 & 0 & \sqrt{2} \\ 0 & 3 & 1 & 0 \\ \sqrt{2} & 0 & 0 & -2\sqrt{2} \end{pmatrix}.$$

(b) The eigenimages are given by

$$\vec{u}_1 \vec{v}_1^T = \frac{1}{\sqrt{5}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} (0 \ 2 \ 0 \ 1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{2\sqrt{5}}{5} & 0 & \frac{\sqrt{5}}{5} \end{pmatrix} \text{ and}$$

$$\vec{u}_2 \vec{v}_2^T = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1 \ 0 \ 3 \ 0) = \begin{pmatrix} \frac{\sqrt{10}}{10} & 0 & \frac{3\sqrt{10}}{10} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$\text{Hence } A = 2\sqrt{5} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{2\sqrt{5}}{5} & 0 & \frac{\sqrt{5}}{5} \end{pmatrix} + \sqrt{10} \begin{pmatrix} \frac{\sqrt{10}}{10} & 0 & \frac{3\sqrt{10}}{10} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$4. \quad (a) \quad f^T f = f f^T = \begin{pmatrix} a^2 + b^2 + c^2 & ab + bc + ca & ab + bc + ca \\ ab + bc + ca & a^2 + b^2 + c^2 & ab + bc + ca \\ ab + bc + ca & ab + bc + ca & a^2 + b^2 + c^2 \end{pmatrix}.$$

Denote  $a^2 + b^2 + c^2$  by  $p$ , and  $ab + bc + ca$  by  $q$ .

$$\begin{aligned} \det(f^T f - \lambda I) &= \begin{vmatrix} p - \lambda & q & q \\ q & p - \lambda & q \\ q & q & p - \lambda \end{vmatrix} \\ &= \begin{vmatrix} p - \lambda & q & q \\ q & p - \lambda & q \\ 0 & \lambda + q - p & p - q - \lambda \end{vmatrix} \\ &= (p - q - \lambda) \begin{vmatrix} p - \lambda & q & q \\ q & p - \lambda & q \\ 0 & -1 & 1 \end{vmatrix} \end{aligned}$$

$$\begin{aligned}
&= (p - q - \lambda) \begin{vmatrix} p - \lambda & 2q & q \\ q & p + q - \lambda & q \\ 0 & 0 & 1 \end{vmatrix} \\
&= (p - q - \lambda) \begin{vmatrix} p - \lambda & 2q \\ q & p + q - \lambda \end{vmatrix} \\
&= (p - q - \lambda)[\lambda^2 - (2p + q)\lambda + p^2 + pq - 2q^2] \\
&= (p - q - \lambda)\left(p + \frac{q}{2} + \frac{3|q|}{2}\right)\left(p + \frac{q}{2} - \frac{3|q|}{2}\right) \\
&= (p - q - \lambda)^2(p + 2q - \lambda) \\
&= (a^2 + b^2 + c^2 - ab - bc - ca - \lambda)^2(a^2 + b^2 + c^2 + 2ab + 2bc + 2ca - \lambda).
\end{aligned}$$

Note that  $p - q = (a - b)^2 + (b - c)^2 + (c - a)^2$  and  $p + 2q = (a + b + c)^2$ .

Suppose  $q \neq 0$ .

If  $p - q \neq 0$ , i.e.  $a, b, c$  not all equal,

for  $\lambda = p - q = a^2 + b^2 + c^2 - ab - bc - ca$ :

$$\left[ \begin{array}{ccc|c} q & q & q & 0 \\ q & q & q & 0 \\ q & q & q & 0 \end{array} \right] \sim \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right],$$

which gives eigenvectors  $\vec{v}_1 = \frac{1}{\sqrt{2}}(1, 0, -1)^T$  and  $\vec{v}_2 = (0, 1, -1)^T$ ;

by Gram-Schmidt orthonormalization,  $\vec{v}_2$  for  $q \neq 0$  is given by

$$\begin{cases} \vec{v}_2 = \vec{v}_2 - \langle \vec{v}_1, \vec{v}_2 \rangle \vec{v}_1 = (0, 1, -1)^T - \frac{1}{2}(1, 0, -1)^T = \frac{1}{2}(-1, 2, -1)^T, \\ \vec{v}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|} = \frac{1}{\sqrt{6}}(-1, 2, -1)^T. \end{cases}$$

Then

$$\begin{aligned}
\vec{u}_1 &= \frac{f\vec{v}_1}{\sqrt{p - q}} = \frac{1}{\sqrt{2(p - q)}} \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \\
&= \frac{1}{\sqrt{2(p - q)}}(a - c, c - b, b - a)^T,
\end{aligned}$$

and

$$\begin{aligned}
\vec{u}_2 &= \frac{f\vec{v}_2}{\sqrt{p - q}} = \frac{1}{\sqrt{6(p - q)}} \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \\
&= \frac{1}{\sqrt{6(p - q)}}(2b - c - a, 2a - b - c, 2c - a - b)^T.
\end{aligned}$$

If  $p + 2q \neq 0$ , i.e.  $a + b + c \neq 0$ , for  $\lambda = p + 2q = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$ ,

$$\begin{aligned}
\left[ \begin{array}{ccc|c} -2q & q & q & 0 \\ q & -2q & q & 0 \\ q & q & -2q & 0 \end{array} \right] &\sim \left[ \begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ -2 & 1 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array} \right] \sim \left[ \begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{array} \right] \\
&\sim \left[ \begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \sim \left[ \begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right],
\end{aligned}$$

which give unit eigenvector  $\vec{v}_3 = \frac{1}{\sqrt{3}}(1, 1, 1)^T$ .

Then

$$\begin{aligned}
\vec{u}_3 &= \frac{f\vec{v}_3}{\sqrt{p + 2q}} = \frac{1}{\sqrt{3(p + 2q)}} \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\
&= \frac{a + b + c}{\sqrt{3(p + 2q)}}(1, 1, 1)^T = \frac{1}{\sqrt{3}}(1, 1, 1)^T.
\end{aligned}$$

Hence if  $q \neq 0$  and  $p - q \neq 0$ , an SVD of  $f$  is given by  $f = U\Sigma V^T$ , where

$$U = \begin{pmatrix} \frac{a-c}{\sqrt{2(p-q)}} & \frac{2b-c-a}{\sqrt{6(p-q)}} & \frac{1}{\sqrt{3}} \\ \frac{c-b}{\sqrt{2(p-q)}} & \frac{2a-b-c}{\sqrt{6(p-q)}} & \frac{1}{\sqrt{3}} \\ \frac{b-a}{\sqrt{2(p-q)}} & \frac{2c-a-b}{\sqrt{6(p-q)}} & \frac{1}{\sqrt{3}} \end{pmatrix}, \Sigma = \begin{pmatrix} \sqrt{p-q} & 0 & 0 \\ 0 & \sqrt{p-q} & 0 \\ 0 & 0 & \sqrt{p+2q} \end{pmatrix}$$

$$\text{and } V = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}.$$

If  $q \neq 0$  and  $p - q = 0$ , then  $p + 2q \neq 0$ . Then since  $\vec{v}_1, \vec{v}_2 \in \text{span}((1, 1, 1)^T)$ , we can use them to form  $U$  as well, i.e.  $\vec{u}_1 = \vec{v}_1, \vec{u}_2 = \vec{v}_2$  and thus  $f = V\Sigma V^T$ , where  $V$  and  $\Sigma$  are

$$\text{the same as above with } \Sigma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sqrt{p+2q} \end{pmatrix}.$$

On the other hand, if  $q = 0$ ,  $\Sigma = \sqrt{p}I_3$  and thus any  $3 \times 3$  unitary matrix  $W$  would satisfy  $f = W\Sigma W^T$ .

- (b) Suppose  $p = 0$ . Then  $a = b = c = 0$  and  $\text{rank}(f) = \text{rank}(\mathbf{0}) = 0$ , and thus  $f$  has no rank-2 approximation.

Suppose  $p \neq 0$  and  $q = 0$ . Then for any orthonormal  $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^3$ ,  $\sqrt{p}(\vec{v}_1\vec{v}_1^T + \vec{v}_2\vec{v}_2^T)$  is a rank-2 approximation to  $f$ . In particular,

$$\sqrt{p}[(1, 0, 0)(1, 0, 0)^T + (0, 1, 0)(0, 1, 0)^T] = \sqrt{p} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\sqrt{p}[(1, 0, 0)(1, 0, 0)^T + (0, 0, 1)(0, 0, 1)^T] = \sqrt{p} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are distinct rank-2 approximations of  $f$ .

Suppose  $p \neq 0$ ,  $q \neq 0$  and  $p - q = 0$ . Then  $f = V\Sigma V^T$  is rank 1 and  $f$  has no rank-2 approximation.

Suppose  $p \neq 0$ ,  $q > 0$  and  $p - q \neq 0$ . Then  $\sqrt{p+2q} > \sqrt{p-q}$ , and thus  $\sqrt{p+2q}\vec{u}_3\vec{v}_3^T + \sqrt{p-q}\vec{u}_1\vec{v}_1^T$  and  $\sqrt{p+2q}\vec{u}_3\vec{v}_3^T + \sqrt{p-q}\vec{u}_2\vec{v}_2^T$  are distinct rank-2 approximations of  $f$ .

Hence for  $f$  to have a unique rank-2 approximation,  $p \neq 0$ ,  $q < 0$  and  $p - q \neq 0$ , i.e.  $a, b, c$  are not all equal and  $ab + bc + ca < 0$ .

$$5. \quad \text{(a)} \quad \tilde{H} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix}.$$

$$f_{\text{Haar}} = \tilde{H}f\tilde{H}^T$$

$$= \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \begin{pmatrix} 5 & 4 & 6 & 6 \\ 6 & 1 & 6 & 3 \\ 1 & 2 & 1 & 5 \\ 6 & 4 & 6 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{pmatrix}$$

$$= \frac{1}{4} \begin{pmatrix} 18 & 11 & 19 & 15 \\ 4 & -1 & 5 & 3 \\ -\sqrt{2} & 3\sqrt{2} & 0 & 3\sqrt{2} \\ -5\sqrt{2} & -2\sqrt{2} & -5\sqrt{2} & 4\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{pmatrix}$$

$$= \frac{1}{4} \begin{pmatrix} 63 & -5 & 7\sqrt{2} & 4\sqrt{2} \\ 11 & -5 & 5\sqrt{2} & 2\sqrt{2} \\ 5\sqrt{2} & -\sqrt{2} & -8 & -6 \\ -8\sqrt{2} & -6\sqrt{2} & -6 & -18 \end{pmatrix} = \begin{pmatrix} \frac{63}{4} & -\frac{5}{4} & \frac{7\sqrt{2}}{4} & \frac{\sqrt{2}}{2} \\ \frac{11}{4} & -\frac{5}{4} & \frac{5\sqrt{2}}{4} & \frac{\sqrt{2}}{2} \\ \frac{5\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & -2 & -\frac{3}{2} \\ -2\sqrt{2} & -\frac{3\sqrt{2}}{2} & -\frac{3}{2} & -\frac{9}{2} \end{pmatrix}.$$

(b) Since  $\tilde{H}$  is unitary, for any  $g \in M_{4 \times 4}(\mathbb{R})$ ,

$$\|\tilde{H}^T g \tilde{H} - f\|_F = \|\tilde{H}^T (g - f_{\text{Haar}}) \tilde{H}\|_F = \|g - f_{\text{Haar}}\|_F.$$

Hence one should choose to discard the entries with smaller absolute values so as to minimize the Frobenius norm of the difference.

$$\text{Hence the matrix that should be kept is either } f'_{\text{Haar}} = \begin{pmatrix} \frac{63}{4} & 0 & \frac{7\sqrt{2}}{4} & 0 \\ \frac{11}{4} & 0 & \frac{5\sqrt{2}}{4} & 0 \\ \frac{5\sqrt{2}}{4} & 0 & -2 & 0 \\ -2\sqrt{2} & -\frac{3\sqrt{2}}{2} & -\frac{3}{2} & -\frac{9}{2} \end{pmatrix},$$

whose reconstructed image is given by  $\tilde{H}^T f'_{\text{Haar}} \tilde{H}$

$$\begin{aligned} &= \frac{1}{16} \begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} 63 & 0 & 7\sqrt{2} & 0 \\ 11 & 0 & 5\sqrt{2} & 0 \\ 5\sqrt{2} & 0 & -8 & 0 \\ -8\sqrt{2} & -6\sqrt{2} & -6 & -18 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \\ &= \frac{1}{16} \begin{pmatrix} 84 & 0 & 4\sqrt{2} & 0 \\ 64 & 0 & 20\sqrt{2} & 0 \\ 36 & -12 & -4\sqrt{2} & -18\sqrt{2} \\ 68 & 12 & 8\sqrt{2} & 18\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \\ &= \frac{1}{16} \begin{pmatrix} 92 & 76 & 84 & 84 \\ 104 & 24 & 64 & 64 \\ 16 & 32 & 12 & 84 \\ 96 & 64 & 92 & 20 \end{pmatrix} = \begin{pmatrix} \frac{23}{4} & \frac{19}{4} & \frac{21}{4} & \frac{21}{4} \\ \frac{13}{2} & \frac{3}{2} & 4 & 4 \\ 1 & 2 & 3 & \frac{21}{4} \\ 6 & 4 & \frac{23}{4} & \frac{5}{4} \end{pmatrix}; \end{aligned}$$

$$\text{or keep } f''_{\text{Haar}} = \begin{pmatrix} \frac{63}{4} & 0 & \frac{7\sqrt{2}}{4} & 0 \\ \frac{11}{4} & 0 & \frac{5\sqrt{2}}{4} & 0 \\ \frac{5\sqrt{2}}{4} & 0 & -2 & -\frac{3}{2} \\ -2\sqrt{2} & -\frac{3\sqrt{2}}{2} & 0 & -\frac{9}{2} \end{pmatrix},$$

whose reconstructed image is given by  $\tilde{H}^T f''_{\text{Haar}} \tilde{H}$

$$\begin{aligned} &= \frac{1}{16} \begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} 63 & 0 & 7\sqrt{2} & 0 \\ 11 & 0 & 5\sqrt{2} & 0 \\ 5\sqrt{2} & 0 & -8 & -6 \\ -8\sqrt{2} & -6\sqrt{2} & 0 & -18 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \\ &= \frac{1}{16} \begin{pmatrix} 84 & 0 & 4\sqrt{2} & -6\sqrt{2} \\ 64 & 0 & 20\sqrt{2} & 6\sqrt{2} \\ 36 & -12 & 2\sqrt{2} & -18\sqrt{2} \\ 68 & 12 & 2\sqrt{2} & 18\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \\ &= \frac{1}{16} \begin{pmatrix} 92 & 76 & 72 & 96 \\ 104 & 24 & 76 & 52 \\ 28 & 20 & 12 & 84 \\ 84 & 76 & 92 & 20 \end{pmatrix} = \begin{pmatrix} \frac{23}{4} & \frac{19}{4} & \frac{9}{4} & 6 \\ \frac{13}{2} & \frac{3}{2} & \frac{19}{4} & \frac{13}{4} \\ \frac{7}{4} & \frac{5}{4} & \frac{3}{4} & \frac{21}{4} \\ \frac{21}{4} & \frac{19}{4} & \frac{23}{4} & \frac{5}{4} \end{pmatrix}. \end{aligned}$$

$$6. \quad (\text{a}) \quad \int_{\mathbb{R}} [H_0(t)]^2 dt = \int_0^1 dt = 1.$$

For any  $p \in \mathbb{N} \setminus \{0\}$  and  $n \in \mathbb{Z} \cap [0, 2^p - 1]$ ,

$$\begin{aligned} \int_{\mathbb{R}} [H_{2^p+n}(t)]^2 dt &= \int_{\frac{n}{2^p}}^{\frac{n+0.5}{2^p}} (2^{\frac{p}{2}})^2 dt + \int_{\frac{n+0.5}{2^p}}^{\frac{n+1}{2^p}} (-2^{\frac{p}{2}})^2 dt \\ &= 2 \cdot \frac{1}{2^{p+1}} \cdot 2^p = 1. \end{aligned}$$

- (b) i. Let  $m \in \mathbb{N} \setminus \{0\}$ . There exists  $p \in \mathbb{N} \cup \{0\}$  and  $n \in \mathbb{Z} \cap [0, 2^p - 1]$  such that  $m = 2^p + n$ . Then

$$\begin{aligned} \langle H_0, H_m \rangle &= \int_{\mathbb{R}} H_0(t) H_{2^p+n}(t) dt \\ &= \int_{\frac{n}{2^p}}^{\frac{n+0.5}{2^p}} 2^{\frac{p}{2}} dt + \int_{\frac{n+0.5}{2^p}}^{\frac{n+1}{2^p}} (-2^{\frac{p}{2}}) dt \\ &= \frac{1}{2^{p+1}} \cdot 2^{\frac{p}{2}} + \frac{1}{2^{p+1}} \cdot (-2^{\frac{p}{2}}) = 0. \end{aligned}$$

- ii. A. Suppose  $p_1 = p_2$ . Then

$$\begin{aligned} \langle H_{m_1}, H_{m_2} \rangle &= \int_{\mathbb{R}} H_{2^{p_1}+n_1}(t) H_{2^{p_1}+n_2}(t) dt \\ &= \int_{\frac{n_1}{2^{p_1}}}^{\frac{n_1+0.5}{2^{p_1}}} 2^{\frac{p_1}{2}} \cdot 0 dt + \int_{\frac{n_1+0.5}{2^{p_1}}}^{\frac{n_1+1}{2^{p_1}}} (-2^{\frac{p_1}{2}}) \cdot 0 dt \\ &\quad + \int_{\frac{n_2}{2^{p_1}}}^{\frac{n_2+0.5}{2^{p_1}}} 0 \cdot 2^{\frac{p_1}{2}} + \int_{\frac{n_2+0.5}{2^{p_1}}}^{\frac{n_2+1}{2^{p_1}}} 0 \cdot (-2^{\frac{p_1}{2}}) dt = 0. \end{aligned}$$

- B. Suppose  $p_1 < p_2$ . Then either

- $2^{p_2-p_1}n_1 \leq n_2 < 2^{p_2-p_1}(n_1+0.5)$  and thus  $\left[\frac{n_2}{2^{p_2}}, \frac{n_2+1}{2^{p_2}}\right] \subseteq \left[\frac{n_1}{2^{p_1}}, \frac{n_1+0.5}{2^{p_1}}\right]$ ;  
or
- $2^{p_2-p_1}(n_1+0.5) \leq n_2 < 2^{p_2-p_1}(n_1+1)$  and thus  $\left[\frac{n_2}{2^{p_2}}, \frac{n_2+1}{2^{p_2}}\right] \subseteq \left[\frac{n_1+0.5}{2^{p_1}}, \frac{n_1+1}{2^{p_1}}\right]$ ;  
or
- $\left[\frac{n_2}{2^{p_2}}, \frac{n_2+1}{2^{p_2}}\right] \cap \left[\frac{n_1+0.5}{2^{p_1}}, \frac{n_1+1}{2^{p_1}}\right] = \emptyset$ .

In any case,  $H_{m_1}$  is constant on  $\left[\frac{n_2}{2^{p_2}}, \frac{n_2+1}{2^{p_2}}\right)$ , and thus denoting the constant by  $c$ ,

$$\begin{aligned} \langle H_{m_1}, H_{m_2} \rangle &= \int_{\mathbb{R}} H_{2^{p_1}+n_1}(t) H_{2^{p_2}+n_2}(t) dt \\ &= c \int_{\frac{n_2}{2^{p_2}}}^{\frac{n_2+0.5}{2^{p_2}}} 2^{\frac{p_2}{2}} dt + c \int_{\frac{n_2+0.5}{2^{p_2}}}^{\frac{n_2+1}{2^{p_2}}} (-2^{\frac{p_2}{2}}) dt \\ &= c \left[ \frac{1}{2^{p_2+1}} \cdot 2^{\frac{p_2}{2}} + \frac{1}{2^{p_2+1}} \cdot (-2^{\frac{p_2}{2}}) \right] = 0. \end{aligned}$$

7. (a)  $\tilde{W} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$ .

$$f_{\text{Walsh}} = \tilde{W} f \tilde{W}^T$$

$$\begin{aligned} &= \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 3 & 2 & 4 & 4 \\ 4 & -2 & 4 & 1 \\ -2 & -1 & -1 & 3 \\ 4 & 1 & 4 & -2 \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \\ &= \frac{1}{4} \begin{pmatrix} 9 & 0 & 11 & 6 \\ -5 & 0 & -5 & -4 \\ -5 & -6 & -5 & 2 \\ -7 & 2 & -5 & 8 \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \\ &= \frac{1}{4} \begin{pmatrix} 26 & 8 & -4 & 14 \\ -14 & -4 & 4 & -6 \\ -14 & 8 & -8 & -6 \\ -2 & 8 & -4 & -22 \end{pmatrix} = \begin{pmatrix} \frac{13}{2} & 2 & -1 & \frac{7}{2} \\ -\frac{7}{2} & -1 & 1 & -\frac{3}{2} \\ -\frac{7}{2} & 2 & -2 & -\frac{3}{2} \\ -\frac{1}{2} & 2 & -1 & -\frac{11}{2} \end{pmatrix}. \end{aligned}$$

- (b) The modified Walsh transform  $f'_{\text{Walsh}}$  is  $\begin{pmatrix} 7 & 2 & -1 & 4 \\ -4 & -1 & 1 & -2 \\ -4 & 2 & -2 & -2 \\ -1 & 2 & -1 & -6 \end{pmatrix}$ , whose reconstructed image is given by

$$\begin{aligned} \tilde{W}^T f'_{\text{Walsh}} \tilde{W} &= \frac{1}{4} \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 7 & 2 & -1 & 4 \\ -4 & -1 & 1 & -2 \\ -4 & 2 & -2 & -2 \\ -1 & 2 & -1 & -6 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \\ &= \frac{1}{4} \begin{pmatrix} 14 & 3 & -1 & 2 \\ 8 & 3 & -3 & 10 \\ -2 & 5 & -3 & -6 \\ 8 & -3 & 3 & 10 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \\ &= \frac{1}{4} \begin{pmatrix} 14 & 8 & 18 & 16 \\ 18 & -8 & 18 & 4 \\ -10 & -4 & -6 & 12 \\ 18 & 4 & 18 & -8 \end{pmatrix} = \begin{pmatrix} 3 & \frac{3}{2} & 5 & \frac{9}{2} \\ \frac{9}{2} & -2 & \frac{9}{2} & 1 \\ -\frac{5}{2} & -1 & -\frac{3}{2} & 3 \\ \frac{9}{2} & 1 & \frac{9}{2} & -2 \end{pmatrix} \end{aligned}$$

8. (a) Note that  $W_0 = \mathbf{1}_{[0,1]}$  and thus  $(W_0)^2 = \mathbf{1}_{[0,1]}$ . Recall that for any  $n \in \mathbb{N} \cup \{0\}$ ,  $W_n$  is defined by the recursive relation:

$$W_{2j+q}(t) = (-1)^{\lfloor \frac{j}{2} \rfloor + q} W_j(2t) + (-1)^{j + \lfloor \frac{j}{2} \rfloor} W_j(2t - 1)$$

for  $j \in \mathbb{N} \cup \{0\}$  and  $q \in \{0, 1\}$ .

Hence for any  $n \in \mathbb{N}$ ,  $(W_n)^2 \equiv \mathbf{1}_{[0,1]}$  and thus

$$\int_{\mathbb{R}} [W_n(t)]^2 dt = \int_0^1 dt = 1.$$

- (b) i. Suppose  $j_1 = j_2$ . Then  $m_1 = 2j_1$  and  $m_2 = 2j_1 + 1$ , and

$$\begin{aligned} \langle W_{m_1}, W_{m_2} \rangle &= \int_{\mathbb{R}} W_{2j_1}(t) W_{2j_1+1}(t) dt \\ &= \int_0^{\frac{1}{2}} (-1)^{\lfloor \frac{j_1}{2} \rfloor} W_{j_1}(2t) \cdot (-1)^{\lfloor \frac{j_1}{2} \rfloor + 1} W_{j_1}(2t) dt \\ &\quad + \int_{\frac{1}{2}}^1 (-1)^{j_1 + \lfloor \frac{j_1}{2} \rfloor} W_{j_1}(2t - 1) \cdot (-1)^{j_1 + \lfloor \frac{j_1}{2} \rfloor} W_{j_1}(2t - 1) dt \\ &= - \int_0^1 [W_{j_1}(u)]^2 d\left(\frac{u}{2}\right) + \int_0^1 [W_{j_1}(v)]^2 d\left(\frac{v-1}{2}\right) \\ &= -\frac{1}{2} \|W_{j_1}\|^2 + \frac{1}{2} \|W_{j_1}\|^2 = 0. \end{aligned}$$

- ii. Suppose  $j_1 < j_2$ . Then

$$\begin{aligned} \langle W_{m_1}, W_{m_2} \rangle &= \int_{\mathbb{R}} W_{2j_1+q_1}(t) W_{2j_2+q_2}(t) dt \\ &= \int_0^{\frac{1}{2}} (-1)^{\lfloor \frac{j_1}{2} \rfloor + q_1} W_{j_1}(2t) \cdot (-1)^{\lfloor \frac{j_2}{2} \rfloor + q_2} W_{j_2}(2t) dt \\ &\quad + \int_{\frac{1}{2}}^1 (-1)^{j_1 + \lfloor \frac{j_1}{2} \rfloor} W_{j_1}(2t - 1) \cdot (-1)^{j_2 + \lfloor \frac{j_2}{2} \rfloor} W_{j_2}(2t - 1) dt \\ &= (-1)^{\lfloor \frac{j_1}{2} \rfloor + \lfloor \frac{j_2}{2} \rfloor + q_1 + q_2} \cdot \frac{1}{2} \int_0^1 W_{j_1}(u) W_{j_2}(u) du \\ &\quad + (-1)^{j_1 + j_2 + \lfloor \frac{j_1}{2} \rfloor + \lfloor \frac{j_2}{2} \rfloor} \cdot \frac{1}{2} \int_0^1 W_{j_1}(v) W_{j_2}(v) dv \\ &= \left[ (-1)^{\lfloor \frac{j_1}{2} \rfloor + \lfloor \frac{j_2}{2} \rfloor + q_1 + q_2} + (-1)^{j_1 + j_2 + \lfloor \frac{j_1}{2} \rfloor + \lfloor \frac{j_2}{2} \rfloor} \right] \langle W_{j_1}, W_{j_2} \rangle = 0 \end{aligned}$$

by the induction hypothesis.



**Remark.** Recall that  $P(m)$  states that

$$\{W_0, \dots, W_m\} \text{ is orthogonal in } (L^2(\mathbb{R}), \langle \cdot, \cdot \rangle).$$

Hence even if we have proven  $P(m)$  to be true for any  $m \in \mathbb{N} \cup \{0\}$ ,

$$\mathcal{W} \text{ is orthogonal in } (L^2(\mathbb{R}), \langle \cdot, \cdot \rangle)$$

has not been directly proven. The subtle difference is easier to observe if we consider the statements

$$\tilde{P}(m) : \{0, \dots, m\} \text{ is finite}$$

and

$$\mathbb{N} \cup \{0\} \text{ is finite,}$$

for which  $\tilde{P}(m)$  being true for any  $m \in \mathbb{N} \cup \{0\}$  does not imply the truthfulness of the second statement. However, since the orthogonality of  $\mathcal{W}$  depends on the orthogonality of pairs of its elements, and each pair of its elements is contained in some  $\{W_0, \dots, W_m\}$ , the induction result suffices.

9. (a)  $W_0 = \mathbf{1}_{[0,1]}$ , so  $W_0(0) = \lim_{t \rightarrow 1^-} W_0(t) = 1$ .

Let  $P(k)$  be the proposition that

$$\lim_{t \rightarrow 1^-} W_k(t) = \begin{cases} W_k(0) & \text{if } k \text{ is even,} \\ -W_k(0) & \text{if } k \text{ is odd.} \end{cases}$$

Suppose  $P(j)$  is true for some  $j \in \mathbb{N} \cup \{0\}$ .

Then for  $q \in \{0, 1\}$ ,

$$\begin{aligned} \lim_{t \rightarrow 1^-} W_{2j+q}(t) &= \lim_{t \rightarrow 1^-} (-1)^{j+q} W_j(2t-1) \\ &= (-1)^{j+q} \lim_{t \rightarrow 1^-} W_j(t) \\ &= \begin{cases} (-1)^q \lim_{t \rightarrow 1^-} W_j(t) & \text{if } j \text{ is even,} \\ (-1)^{q+1} \lim_{t \rightarrow 1^-} W_j(t) & \text{if } j \text{ is odd} \end{cases} \\ &= (-1)^q W_j(0) = \begin{cases} W_j(0) & \text{if } q = 0, \\ -W_j(0) & \text{if } q = 1. \end{cases} \end{aligned}$$

Hence  $P(2j)$  and  $P(2j+1)$  are also true. By induction,  $P(k)$  is true for any  $k \in \mathbb{N} \cup \{0\}$ .

- (b)  $W_0 \equiv \mathbf{1}_{[0,1]}$  and thus has 0 zero-crossings on  $(0,1)$ .

Let  $P(k)$  be the proposition that  $W_k$  has  $k$  zero-crossings on  $(0,1)$ .

Suppose  $P(j)$  is true for some  $j \in \mathbb{N} \cup \{0\}$ . Then a direct observation is that both  $W_{2j}$  and  $W_{2j+1}$  have  $j$  zero-crossings on each of  $(0, \frac{1}{2})$  and  $(\frac{1}{2}, 1)$ ; it remains to verify whether  $\frac{1}{2}$  is a zero-crossing of theirs.

For  $q \in \{0, 1\}$ ,

$$\begin{aligned} \lim_{t \rightarrow \frac{1}{2}^-} W_{2j+q}(t) &= (-1)^{\lfloor \frac{j}{2} \rfloor + q} \lim_{t \rightarrow 1^-} W_j(t) \\ &= \begin{cases} (-1)^{\lfloor \frac{j}{2} \rfloor + q} W_j(0) & \text{if } j \text{ is even,} \\ (-1)^{\lfloor \frac{j}{2} \rfloor + q + 1} W_j(0) & \text{if } j \text{ is odd} \end{cases} \\ &= \begin{cases} (-1)^{j+q} W_{2j+q}(\frac{1}{2}) & \text{if } j \text{ is even,} \\ (-1)^{j+q+1} W_{2j+q}(\frac{1}{2}) & \text{if } j \text{ is odd} \end{cases} \\ &= (-1)^q W_{2j+q} \left( \frac{1}{2} \right) = \begin{cases} \lim_{t \rightarrow \frac{1}{2}^+} W_{2j+q}(t) & \text{if } q = 0, \\ - \lim_{t \rightarrow \frac{1}{2}^+} W_{2j+q}(t) & \text{if } q = 1. \end{cases} \end{aligned}$$

Hence  $\frac{1}{2}$  is a zero-crossing of  $W_{2j+q}$  if  $q = 1$  and is not if  $q = 0$ , and thus  $W_{2j}$  has  $2j$  zero-crossings on  $(0,1)$  while  $W_{2j+1}$  has  $2j+1$ .

By induction,  $P(k)$  is true for any  $k \in \mathbb{N} \cup \{0\}$ .

10. (a) The idea is basically the same with Q6(b)(ii)(B):

Let  $\emptyset \neq M \subseteq \mathbb{N} \setminus \{0\}$  and let  $m^* = \max M$ .

Then  $\prod_{m \in M \setminus \{m^*\}} R_m$  is constant on each period of  $R_{m^*}$ , i.e.

$$\prod_{m \in M \setminus \{m^*\}} R_m \equiv c_k \in \{\pm 1\} \text{ on each } \left( \frac{k}{2^{m^*-1}}, \frac{k+1}{2^{m^*-1}} \right), k = 0, 1, \dots, 2^{m^*-1} - 1.$$

Hence

$$\begin{aligned} \int_{\mathbb{R}} \prod_{m \in M} R_m &= \sum_{k=0}^{2^{m^*}-1} \int_{\frac{k}{2^{m^*-1}}}^{\frac{k+1}{2^{m^*-1}}} c_k R_{m^*}(t) dt \\ &= \sum_{k=0}^{2^{m^*}-1} c_k \left( \int_{\frac{2k}{2^{m^*}}}^{\frac{2k+1}{2^{m^*}}} dt - \int_{\frac{2k+1}{2^{m^*}}}^{\frac{2k+2}{2^{m^*}}} dt \right) = 0. \end{aligned}$$

(b) Let  $M$  be a finite subset of  $\mathbb{N} \setminus \{0\}$ .

For any  $m \in \mathbb{N} \setminus \{0\}$ ,  $R_m^2 \equiv \mathbf{1}_{[0,1)}$  except at finitely many points. Hence  $\left( \prod_{m \in M} R_m \right)^2 \equiv \mathbf{1}_{[0,1)}$  except at finitely many points, and

$$\int_{\mathbb{R}} \left( \prod_{m \in M} R_m \right)^2 = \int_0^1 dt = 1.$$

Let  $M_1$  and  $M_2$  be distinct finite subsets of  $\mathbb{N} \setminus \{0\}$ . Then  $(M_1 \setminus M_2) \sqcup (M_2 \setminus M_1) \neq \emptyset$ . Except at finitely many points,

$$\begin{aligned} \prod_{m \in M_1} R_m \cdot \prod_{n \in M_2} R_n &= \prod_{m \in M_1 \cap M_2} R_m^2 \cdot \prod_{n \in M_1 \setminus M_2} R_n \cdot \prod_{p \in M_2 \setminus M_1} R_p \\ &= \prod_{m \in (M_1 \setminus M_2) \sqcup (M_2 \setminus M_1)} R_m, \end{aligned}$$

and thus by the result of (a),

$$\int_{\mathbb{R}} \left( \prod_{m \in M_1} R_m \cdot \prod_{n \in M_2} R_n \right) = \int_{\mathbb{R}} \prod_{m \in (M_1 \setminus M_2) \sqcup (M_2 \setminus M_1)} R_m = 0.$$

$$11. \text{ (a) } U = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix}.$$

$$\hat{f} = UfU$$

$$\begin{aligned} &= \frac{1}{16} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix} \begin{pmatrix} 3 & 2 & 4 & 4 \\ 4 & -3 & 4 & 0 \\ -2 & -1 & -2 & 3 \\ 4 & 1 & 4 & -2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix} \\ &= \frac{1}{16} \begin{pmatrix} 9 & -1 & 10 & 5 \\ 5 & 3+4j & 6 & 1-2j \\ -7 & 3 & -6 & 9 \\ 5 & 3-4j & 6 & 1+2j \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix} \\ &= \frac{1}{16} \begin{pmatrix} 23 & -1+6j & 15 & -1-6j \\ 15+2j & 5-2j & 7-2j & -7+2j \\ -1 & -1+6j & -25 & -1-6j \\ 15-2j & -7-2j & 7+2j & 5+2j \end{pmatrix}. \end{aligned}$$

(b) The submatrix of  $\hat{f}$  formed by the three frequencies closest to 0 is

$$\hat{f}' = \frac{1}{16} \begin{pmatrix} 23 & 1+6j & 0 & -1-6j \\ 15+2j & 5-2j & 0 & -7+2j \\ 0 & 0 & 0 & 0 \\ 15-2j & -7-2j & 0 & 5+2j \end{pmatrix},$$

whose reconstructed image is

$$\begin{aligned} (4\bar{U})\hat{f}'(4\bar{U}) &= \frac{1}{16} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{pmatrix} \begin{pmatrix} 23 & 1+6j & 0 & -1-6j \\ 15+2j & 5-2j & 0 & -7+2j \\ 0 & 0 & 0 & 0 \\ 15-2j & -7-2j & 0 & 5+2j \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{pmatrix} \\ &= \frac{1}{16} \begin{pmatrix} 53 & -3+2j & 0 & -3-2j \\ 19 & -1+18j & 0 & -1-18j \\ -7 & 1+10j & 0 & 1-10j \\ 27 & -1-6j & 0 & -1+6j \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{pmatrix} \\ &= \frac{1}{16} \begin{pmatrix} 47 & 49 & 59 & 57 \\ 17 & -17 & 21 & 55 \\ -5 & -27 & -9 & 13 \\ 25 & 39 & 29 & 15 \end{pmatrix}. \end{aligned}$$

12. (a) i. Refer to DFT of convolution of **Further properties of DFT** in Section 4.  
ii.

$$\begin{aligned} iDFT(MN\hat{f} \odot \hat{g})(k, l) &= MN \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \hat{f}(m, n) \hat{g}(m, n) e^{2\pi j(\frac{mk}{M} + \frac{nl}{N})} \\ &= \frac{1}{MN} \sum_{m, k', k''=0}^{M-1} \sum_{n, l', l''=0}^{N-1} f(k', l') g(k'', l'') e^{2\pi j(\frac{m(k-k'-k'')}{M} + \frac{n(l-l'-l'')}{N})} \\ &= \sum_{k', k''=0}^{M-1} \sum_{l', l''=0}^{N-1} f(k', l') g(k'', l'') \mathbf{1}_{M\mathbb{Z}}(k - k' - k'') \mathbf{1}_{N\mathbb{Z}}(l - l' - l'') \\ &= \sum_{k', k''=0}^{M-1} \sum_{l', l''=0}^{N-1} f(k', l') g(k'', l'') [\delta(k - k' - k'') + \delta(k - k' - k'' + M)] \\ &\quad [\delta(l - l' - l'') + \delta(l - l' - l'' + N)] \\ &= \sum_{k'=0}^{M-1} \sum_{l'=0}^{N-1} f(k', l') g(k - k', l - l') = f * g(k, l). \end{aligned}$$

(b) Withheld until the due date of Assignment 3.

(c) i.

$$\begin{aligned} \hat{f}(m, n) &= \frac{1}{N^2} \sum_{k, l=0}^{N-1} \tilde{f}(k, l) e^{-2\pi j \frac{mk+nl}{N}} \\ &= \frac{1}{N^2} \sum_{k, l=0}^{N-1} f(l, -k) e^{-2\pi j \frac{mk+nl}{N}}, \end{aligned}$$

whereas

$$\begin{aligned}
\tilde{f}(m, n) &= \hat{f}(n, -m) \\
&= \frac{1}{N^2} \sum_{k, l=0}^{N-1} f(k, l) e^{-2\pi j \frac{nk - ml}{N}} \\
&= \frac{1}{N^2} \sum_{l'=0}^{N-1} \sum_{k'=1-N}^0 f(l', -k') e^{-2\pi j \frac{mk' + nl'}{N}} \\
&= \frac{1}{N^2} \sum_{l'=0}^{N-1} \left( f(l', 0) e^{-2\pi j \frac{nl'}{N}} + \sum_{k'=1-N}^0 f(l', -k') e^{-2\pi j \frac{mk' + nl'}{N}} \right) \\
&= \frac{1}{N^2} \sum_{k', l'=0}^{N-1} f(l', -k') e^{-2\pi j \frac{mk' + nl'}{N}} = \hat{f}(m, n).
\end{aligned}$$

ii.

$$\begin{aligned}
iDFT(\tilde{f})(k, l) &= \sum_{m, n=0}^{N-1} \tilde{f}(m, n) e^{2\pi j \frac{mk + nl}{N}} \\
&= \sum_{m, n=0}^{N-1} \hat{f}(n, -m) e^{2\pi j \frac{mk + nl}{N}} \\
&= \sum_{m'=0}^{N-1} \sum_{n'=1-N}^0 \hat{f}(m', n') e^{2\pi j \frac{-n'k + m'l}{N}} \\
&= \sum_{m'=0}^{N-1} \left( \hat{f}(m', 0) e^{2\pi j \frac{m'l}{N}} + \sum_{n'=1-N}^{-1} \hat{f}(m', n' + N) e^{2\pi j \frac{-n'k + m'l}{N}} \right) \\
&= \sum_{m', n'=0}^{N-1} \hat{f}(m', n') e^{2\pi j \frac{m'l - n'k}{N}} = f(l, -k).
\end{aligned}$$

(d) WLOG assume  $k_0 \in \mathbb{Z} \cap [0, M-1]$  and  $l_0 \in \mathbb{Z} \cap [0, N-1]$ .

i. Refer to DFT of a shifted image of **Further properties of DFT** in Section 4.

ii.

$$\begin{aligned}
iDFT(e^{-2\pi j (\frac{k_0 m}{M} + \frac{l_0 n}{N})} \hat{f})(k, l) &= \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \hat{f}(m, n) e^{2\pi j (\frac{m(k-k_0)}{M} + \frac{n(l-l_0)}{N})} \\
&= f(k - k_0, l - l_0).
\end{aligned}$$

(e) i.

$$\begin{aligned}
\tilde{f}(m, n) &= \hat{f}(m - m_0, n - n_0) \\
&= \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} f(k, l) e^{-2\pi j (\frac{k(m-m_0)}{M} + \frac{l(n-n_0)}{N})} \\
&= DFT(e^{2\pi j (\frac{m_0 k}{M} + \frac{n_0 l}{N})} f)(m, n).
\end{aligned}$$

ii.

$$\begin{aligned}
iDFT(\tilde{f})(k, l) &= \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \tilde{f}(m, n) e^{2\pi j (\frac{mk}{M} + \frac{nl}{N})} \\
&= \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \hat{f}(m - m_0, n - n_0) e^{2\pi j (\frac{mk}{M} + \frac{nl}{N})} \\
&= \sum_{m'=-m_0}^{M-1-m_0} \sum_{n'=-n_0}^{N-1-n_0} \hat{f}(m', n') e^{2\pi j (\frac{(m'+m_0)k}{M} + \frac{(n'+n_0)l}{N})} \\
&= e^{2\pi j (\frac{m_0 k}{M} + \frac{n_0 l}{N})} f(k, l).
\end{aligned}$$

13. (a) Note that for any  $c \in \mathbb{Z}$ ,  $\cos \pi k = (-1)^k$  and thus

$$\cos \pi k + \cos(-\pi k) = (-1)^k + (-1)^{-k} = 0.$$

$$\begin{aligned} & iEDCT(EDCT(f))(k, l) \\ &= \frac{1}{MN} \sum_{m, k'=0}^{M-1} \sum_{n, l'=0}^{N-1} [2 - \mathbf{1}_{M\mathbb{Z}}(m)][2 - \mathbf{1}_{N\mathbb{Z}}(n)] f(k', l') \\ & \quad \cos \frac{\pi m(2k+1)}{2M} \cos \frac{\pi m(2k'+1)}{2M} \cos \frac{\pi n(2l+1)}{2N} \cos \frac{\pi n(2l'+1)}{2N} \\ &= \frac{1}{4MN} \sum_{k'=0}^{M-1} \sum_{l'=0}^{N-1} \sum_{m'=1-M}^{M-1} \sum_{n'=1-N}^{N-1} f(k', l') \\ & \quad \left( \cos \frac{\pi m(k+k'+1)}{M} + \cos \frac{\pi m(k-k')}{M} \right) \left( \cos \frac{\pi n(l+l'+1)}{N} + \cos \frac{\pi n(l-l')}{N} \right) \\ &= \frac{1}{16MN} \sum_{k'=0}^{M-1} \sum_{l'=0}^{N-1} \sum_{m'=-M}^{M-1} \sum_{n'=-N}^{N-1} f(k', l') \\ & \quad \left( e^{2\pi j \frac{m(k+k'+1)}{2M}} + e^{-2\pi j \frac{m(k+k'+1)}{2M}} + e^{2\pi j \frac{m(k-k')}{2M}} + e^{-2\pi j \frac{m(k-k')}{2M}} \right) \\ & \quad \left( e^{2\pi j \frac{n(l+l'+1)}{2N}} + e^{-2\pi j \frac{n(l+l'+1)}{2N}} + e^{2\pi j \frac{n(l-l')}{2N}} + e^{-2\pi j \frac{n(l-l')}{2N}} \right) \\ &= \sum_{k'=0}^{M-1} \sum_{l'=0}^{N-1} f(k', l') [\mathbf{1}_{2M\mathbb{Z}}(k+k'+1) + \mathbf{1}_{2M\mathbb{Z}}(k-k')] [\mathbf{1}_{2N\mathbb{Z}}(l+l'+1) + \mathbf{1}_{2N\mathbb{Z}}(l-l')] \\ &= f(k, l). \end{aligned}$$

- (b) Assuming  $f$  is symmetrically extended about  $x = 0$  and  $y = 0$ ,

$$\begin{aligned} & iODCT(ODCT(f))(k, l) \\ &= \frac{1}{(2M-1)(2N-1)} \sum_{m, k'=0}^{M-1} \sum_{n, l'=0}^{N-1} [2 - \mathbf{1}_{M\mathbb{Z}}(k)][2 - \mathbf{1}_{N\mathbb{Z}}(l)][2 - \mathbf{1}_{M\mathbb{Z}}(m)][2 - \mathbf{1}_{N\mathbb{Z}}(n)] \\ & \quad f(k', l') \cos \frac{2\pi mk}{2M-1} \cos \frac{2\pi mk'}{2M-1} \cos \frac{2\pi nl}{2N-1} \cos \frac{2\pi nl'}{2N-1} \\ &= \frac{1}{4(2M-1)(2N-1)} \sum_{m, k'=1-M}^{M-1} \sum_{n, l'=1-N}^{N-1} f(k', l') \\ & \quad \left( \cos \frac{2\pi m(k+k')}{2M-1} + \cos \frac{2\pi m(k-k')}{2M-1} \right) \left( \cos \frac{2\pi n(l+l')}{2N-1} + \cos \frac{2\pi n(l-l')}{2N-1} \right) \\ &= \frac{1}{16(2M-1)(2N-1)} \sum_{m, k'=1-M}^{M-1} \sum_{n, l'=1-N}^{N-1} f(k', l') \\ & \quad \left( e^{2\pi j \frac{m(k+k')}{2M-1}} + e^{-2\pi j \frac{m(k+k')}{2M-1}} + e^{2\pi j \frac{m(k-k')}{2M-1}} + e^{-2\pi j \frac{m(k-k')}{2M-1}} \right) \\ & \quad \left( e^{2\pi j \frac{n(l+l')}{2N-1}} + e^{-2\pi j \frac{n(l+l')}{2N-1}} + e^{2\pi j \frac{n(l-l')}{2N-1}} + e^{-2\pi j \frac{n(l-l')}{2N-1}} \right) \\ &= \frac{1}{4} \sum_{k'=1-M}^{M-1} \sum_{l'=1-N}^{N-1} f(k', l') \\ & \quad [\mathbf{1}_{(2M-1)\mathbb{Z}}(k+k') + \mathbf{1}_{(2M-1)\mathbb{Z}}(k-k')] [\mathbf{1}_{(2N-1)\mathbb{Z}}(l+l') + \mathbf{1}_{(2N-1)\mathbb{Z}}(l-l')] \\ &= \frac{1}{4} [f(-k, -l) + f(k, -l) + f(-k, l) + f(k, l)] = f(k, l). \end{aligned}$$

(c) Note that for any  $c \in \mathbb{Z}$ ,  $\sin c\pi = 0$ .

$$\begin{aligned}
& iEDST(EDST(f))(k, l) \\
&= \frac{1}{MN} \sum_{m, k'=0}^{M-1} \sum_{n, l'=0}^{N-1} [2 - \mathbf{1}_{M\mathbb{Z}}(m)][2 - \mathbf{1}_{N\mathbb{Z}}(n)] f(k', l') \\
& \sin \frac{\pi m(2k+1)}{2M} \sin \frac{\pi m(2k'+1)}{2M} \sin \frac{\pi n(2l+1)}{2N} \sin \frac{\pi n(2l'+1)}{2N} \\
&= \frac{1}{4MN} \sum_{k'=0}^{M-1} \sum_{l'=0}^{N-1} \sum_{m=-M}^{M-1} \sum_{n=-N}^{N-1} f(k', l') \\
& \left( \cos \frac{\pi m(k+k'+1)}{M} - \cos \frac{\pi m(k-k')}{M} \right) \left( \cos \frac{\pi n(l+l'+1)}{N} - \cos \frac{\pi n(l-l')}{N} \right) \\
&= \frac{1}{16MN} \sum_{k'=0}^{M-1} \sum_{l'=0}^{N-1} \sum_{m=-M}^{M-1} \sum_{n=-N}^{N-1} f(k', l') \\
& \left( e^{2\pi j \frac{m(k+k'+1)}{2M}} + e^{-2\pi j \frac{m(k+k'+1)}{2M}} - e^{2\pi j \frac{m(k-k')}{2M}} - e^{-2\pi j \frac{m(k-k')}{2M}} \right) \\
& \left( e^{2\pi j \frac{n(l+l'+1)}{2N}} + e^{-2\pi j \frac{n(l+l'+1)}{2N}} - e^{2\pi j \frac{n(l-l')}{2N}} - e^{-2\pi j \frac{n(l-l')}{2N}} \right) \\
&= \sum_{k'=0}^{M-1} \sum_{l'=0}^{N-1} f(k', l') [\mathbf{1}_{2M\mathbb{Z}}(k+k'+1) - \mathbf{1}_{2M\mathbb{Z}}(k-k')] [\mathbf{1}_{2N\mathbb{Z}}(l+l'+1) - \mathbf{1}_{2N\mathbb{Z}}(l-l')] \\
&= f(k, l).
\end{aligned}$$

(d)

$$\begin{aligned}
& iODST(ODST(f))(k, l) \\
&= \frac{16}{(2M+1)(2N+1)} \sum_{k'=0}^{M-1} \sum_{l'=0}^{N-1} \sum_{m=1}^M \sum_{n=1}^N f(k', l') \\
& \sin \frac{2\pi m(k+1)}{2M+1} \sin \frac{2\pi m(k'+1)}{2M+1} \sin \frac{2\pi n(l+1)}{2N+1} \sin \frac{2\pi n(l'+1)}{2N+1} \\
&= \frac{1}{(2M+1)(2N+1)} \sum_{k'=0}^{M-1} \sum_{l'=0}^{N-1} \sum_{m=-M}^M \sum_{n=-N}^N f(k', l') \\
& \left( \cos \frac{2\pi m(k+k'+2)}{2M+1} - \cos \frac{2\pi m(k-k')}{2M+1} \right) \left( \cos \frac{2\pi n(l+l'+2)}{2N+1} - \cos \frac{2\pi n(l-l')}{2N+1} \right) = \frac{1}{4(2M+1)(2N+1)} \\
& \left( e^{2\pi j \frac{m(k+k'+2)}{2M+1}} + e^{-2\pi j \frac{m(k+k'+2)}{2M+1}} - e^{2\pi j \frac{m(k-k')}{2M+1}} - e^{-2\pi j \frac{m(k-k')}{2M+1}} \right) \\
& \left( e^{2\pi j \frac{n(l+l'+2)}{2N+1}} + e^{-2\pi j \frac{n(l+l'+2)}{2N+1}} - e^{2\pi j \frac{n(l-l')}{2N+1}} - e^{-2\pi j \frac{n(l-l')}{2N+1}} \right) \\
&= \frac{1}{4(2M+1)(2N+1)} \sum_{k'=0}^{M-1} \sum_{l'=0}^{N-1} f(k', l') \\
& [2(2M+1)\mathbf{1}_{(2M+1)\mathbb{Z}}(k+k'+2) - 2(2M+1)\mathbf{1}_{(2M+1)\mathbb{Z}}(k-k')] \\
& [2(2N+1)\mathbf{1}_{(2N+1)\mathbb{Z}}(l+l'+2) - 2(2N+1)\mathbf{1}_{(2N+1)\mathbb{Z}}(l-l')] \\
&= \sum_{k'=0}^{M-1} \sum_{l'=0}^{N-1} f(k', l') \delta(k-k') \delta(l-l') = f(k, l).
\end{aligned}$$