Lecture 8:

Recall:

Note:

(Spatial domain) $\quad I * g$ (Linear filtering: Linear combination of neighborhood pixel values)

$$\downarrow \text{DFT}$$

(Frequency domain) $\quad MN\, \underbrace{\hat{I} \odot}_{\text{pixel-wise multiplication}} \hat{g}$ (Modifying the Fourier coefficients by multiplication)

Observation:

1. When k and l are close to $N/2$, $\hat{F}\left(\frac{N}{2}+\underset{\approx N}{k}, \frac{N}{2}+\underset{\approx N}{l}\right)$ is associated to $e^{+j\frac{2\pi}{N}\left(\left(\frac{N}{2}+k\right)m+\left(\frac{N}{2}+l\right)n\right)}$

$$\approx e^{j\frac{2\pi}{N}(k'm+l'n)} \quad \text{where } (k',l') \approx (0,0)$$

$$= \cos\left(\frac{2\pi}{N}(k'm+l'n)\right) + i\sin\left(\frac{2\pi}{N}(k'm+l'n)\right)$$

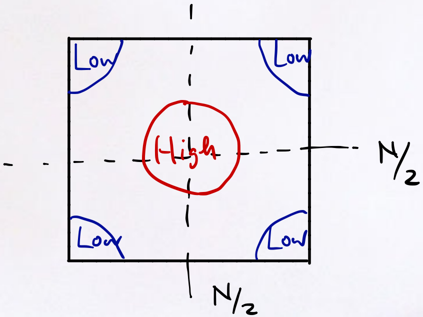
Low-frequency if $(k,l) \approx (0,0)$

$\therefore$ Fourier coefficients at the bottom right are associated to low frequency components!

2. Similarly, we can check that Fourier coefficients at the 4 corners are associated to low frequency components.

3. Fourier coefficients in the middle are associated to high-frequency components

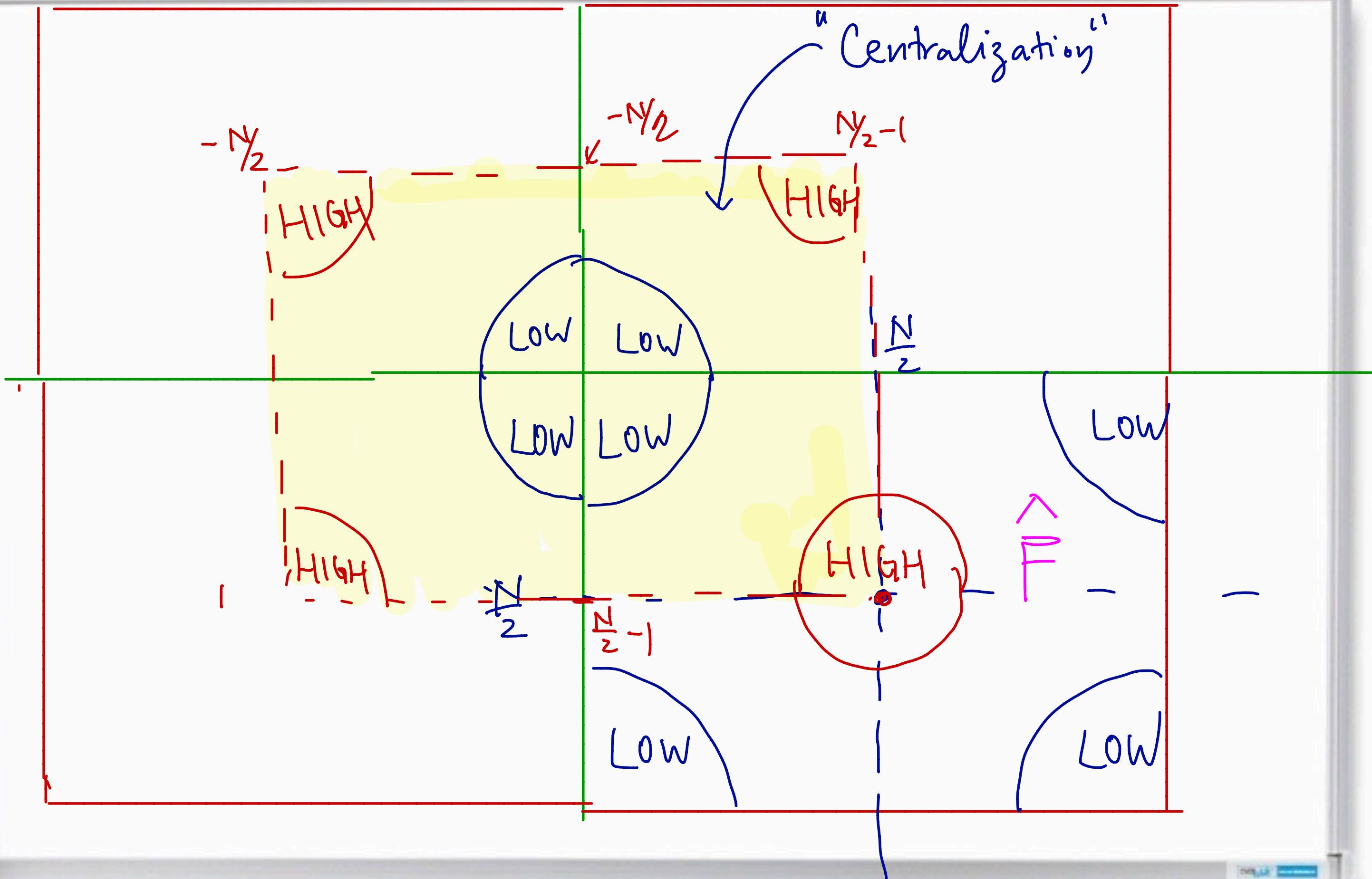
$$e^{j\frac{2\pi}{N}\left(\frac{N}{2}m+\frac{N}{2}n\right)} = e^{j\pi(m+n)} = (-1)^{m+n}$$



!!

$\therefore$ High-pass filtering = Remove coefficients at 4 corners

Low-pass filtering = Remove coefficients at the center


"Centralization"
-N/2
-N/2
N/2-1
HIGH
HIGH
LOW
LOW
LOW
LOW
N/2
LOW
HIGH
HIGH
$\hat{F}$
N/2
N/2-1
LOW
LOW

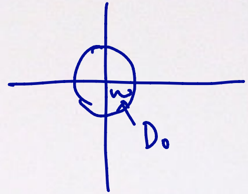
Example of Low-pass filters for image denoising

Assume that we work on the centered spectrum!

That is, consider $\hat{F}(u,v)$ where $-N/2 \le u \le N/2 - 1$, $-N/2 \le v \le N/2 - 1$.

1 Ideal low pass filter (ILPF):

$$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) := u^2 + v^2 \le D_0^2 \\ 0 & \text{if } D(u,v) > D_0^2 \end{cases}$$

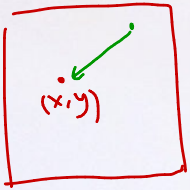


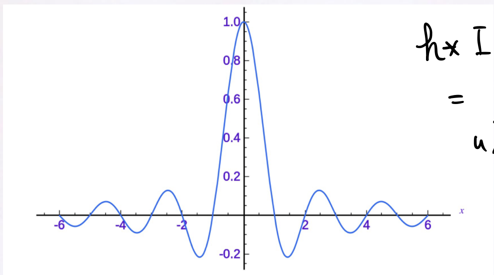
In 1-dim cross-section, $\boxed{\overset{h}{\overset{\shortparallel}{\mathcal{F}^{-1}(H(u,v))}}}$ looks like:

$$I \to \hat{I} \to \hat{I} \odot H$$

$$\downarrow \text{iDFT}$$

$$\tilde{I}$$





$$h * I(x,y) = \sum_{u,v} h(x-u, y-v)\, I(u,v)$$

every pixel values of I has an effect on $h * I(x,y)$!!

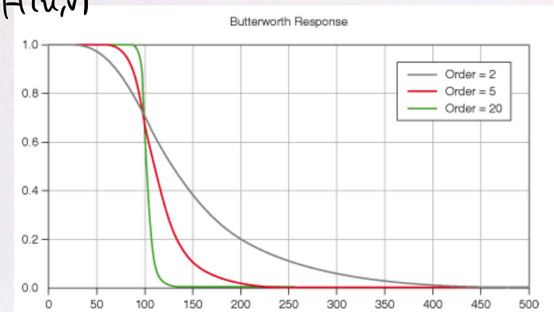
Good: Simple

Bad : Produce ringing effect!

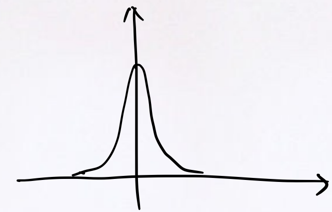
2. Butterworth low-pass filter (BLPF) of order n ($n \geq 1$ integer) :

$$H(u,v) = \frac{1}{1 + \left(D(u,v)/D_0\right)^{2n}}$$

$H(u,v)$ in 1-dim



$\mathcal{F}^{-1}(H(u,v))$ in 1-dim



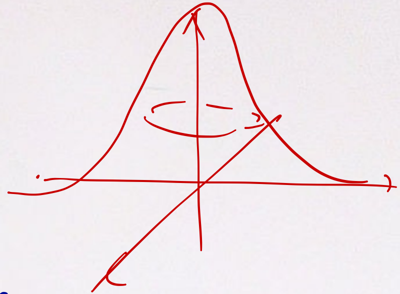
Good: Produce less / no visible ringing effect if n is carefully chosen!!

3. Gaussian low-pass filter

$u^2 + v^2$

$$H(u,v) = \exp\left(-\frac{D(u,v)}{2\sigma^2}\right)$$

σ = spread of the Gaussian function



F.T. of Gaussian is also Gaussian!!

Good: No visible ringing effect!!

Examples for high-pass filtering for feature extraction

1. Ideal high-pass filter: (IHPF)

$$H(u,v) = \begin{cases} 0 & \text{if } D(u,v) \le D_0^2 \\ 1 & \text{if } D(u,v) > D_0^2 \end{cases}$$

Bad: Produce ringing

2. Butterworth high-pass filter:

$$H(u,v) = \frac{1}{1 + \left(\frac{D_0^2}{D(u,v)}\right)^n}$$

$(H(u,v) = 0 \text{ if } D(u,v) = 0)$

Choose the right n

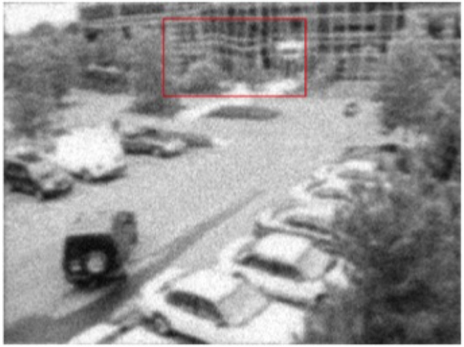
Good: Less ringing

3. Gaussian high-pass filter

$$H(u,v) = 1 - e^{-\left(\frac{D(u,v)}{2\sigma^2}\right)}$$

Good: No visible ringing!

Image deblurring



Atmospheric turbulence

Motion Blur

Speeding problem

Image deblurring in the frequency domain:

Mathematical formulation of image blurring

Let g be the observed (blurry) image.

Let f be the original (good) image.

Model g as: $g = H(f) + n$

where H is the degradation function/operator and n is the additive noise.

Assumption on H:

1. H is position invariant:

   Let $g(x,y) = H(f)(x,y)$ and let $\tilde{f}(x,y) := f(x-\alpha, y-\beta)$.

   Then: $H(\tilde{f})(x,y) = g(x-\alpha, y-\beta)$

2. Linear: $H(f_1 + f_2) = H(f_1) + H(f_2)$

   $H(\alpha f) = \alpha H(f)$ where α is a scalar multiplication.

3. Linearity can be extended to integral:

$$H\left(\iint \alpha(u,v) f(x-u, y-v)\, du\, dv\right) = \iint \alpha(u,v) H(f)(x-u, y-v)\, du\, dv$$

With the above assumption, consider an impluse signal:

$$\delta(x,y) = \begin{cases} 1 & \text{if } (x,y) = (0,0) \\ 0 & \text{if } (x,y) \neq (0,0) \end{cases}$$

Then: $f(x,y) = f * \delta(x,y) = \sum_{\alpha=-M/2}^{M/2-1} \sum_{\beta=-N/2}^{N/2-1} f(\alpha,\beta)\,\delta(x-\alpha, y-\beta)$

$$\begin{aligned} \therefore\quad g(x,y) &= H(f)(x,y) \\ &= \sum_{\alpha=-M/2}^{M/2-1} \sum_{\beta=-N/2}^{N/2-1} f(\alpha,\beta)\, H(\delta)(x-\alpha, y-\beta) \quad \text{(by linearity and position-invariant)} \\ &= \sum_{\alpha=-M/2}^{M/2-1} \sum_{\beta=-N/2}^{N/2-1} f(\alpha,\beta)\, h(x-\alpha, y-\beta) \quad \text{where } h(x,y) = H(\delta)(x,y) \\ &= f * h(x,y) \end{aligned}$$

$\therefore$ With the above assumption,

Degradation / Blur = Convolution

Remark:

1. h is called the point spread function

2. $\therefore g(x,y) = h * f(x,y) + n(x,y)$

In the frequency domain,

$$G(u,v) = c\,H(u,v)\,F(u,v) + N(u,v)$$

constant

$\therefore$ Deblurring can be done by:

Compute: $F(u,v) \approx \dfrac{G(u,v)}{cH(u,v)}$ — from observed image — from known degradation

Obtain: $f(x,y) = DFT^{-1}(F(u,v))$

(Does NOT work very well due to noise!)

Examples of degradation function $H(u,v)$

1. Atmospheric turbulence blur:

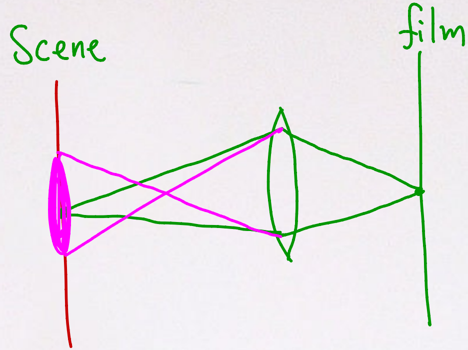
$$H(u,v) = e^{-k(u^2+v^2)^{5/6}}$$

where k = degree of turbulence

$k = 0.0025$ (severe)

$k = 0.001$ (mild)

$k = 0.00025$ (low turbulence)



2. Out of focus blur:

In the frequency domain, define $H(u,v)$ as the DFT of

$$h(x,y) = \begin{cases} 1 & \text{if } x^2+y^2 \le D_0^2 \\ 0 & \text{otherwise} \end{cases}$$

In some situations, a simple model:

$$\cancel{H(u,v) = \begin{cases} 1 & \text{if } u^2+v^2 \le D_0^2 \\ 0 & \text{otherwise} \end{cases}}$$

(Usually not accurate)

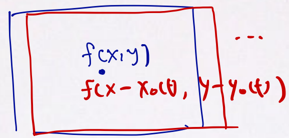
3. Uniform Linear Motion Blur

Assume $f(x,y)$ undergoes planar motion during acquisition.
(original)

(displacement)

Let $(x_0(t), y_0(t))$ be the motion components in the x- and y-directions
time

Let T be the total exposure time.



The observed image is given by:

$$g(x,y) = \int_0^T f(x - x_0(t), y - y_0(t))\, dt$$

Now, let $G(u,v) = DFT(g)(u,v)$, then:

$$G(u,v) = \frac{1}{N^2} \sum_x \sum_y g(x,y)\, e^{-j\frac{2\pi}{N}(ux+vy)}$$

$$= \frac{1}{N^2} \sum_x \sum_y \int_0^T f(x - x_0(t), y - y_0(t))\, dt\; e^{-j\frac{2\pi}{N}(ux+vy)}$$

$$= \int_0^T \left(\sum_x \sum_y f(x - x_0(t), y - y_0(t))\, e^{-j\frac{2\pi}{N}(ux+vy)} \right) dt$$

Recall that $\mathrm{DFT}(f(x-x_0, y-y_0)) = F(u,v)\, e^{-j\frac{2\pi}{N}(u x_0(t) + v y_0(t))}$.

We have: $$G(u,v) = \int_0^T \left[F(u,v)\, e^{-j\frac{2\pi}{N}(u x_0(t) + v y_0(t))} \right] dt$$

$$= F(u,v) \int_0^T e^{-j\frac{2\pi}{N}(u x_0(t) + v y_0(t))}\, dt$$

$$= F(u,v)\, H(u,v)$$

$\therefore$ Degradation function in the frequency domain is given by:

$$H(u,v) = \int_0^T e^{-j\frac{2\pi}{N}(u x_0(t) + v y_0(t))}\, dt$$

(Speeding problem !!)

Example: Suppose the camera is moving left horizontally with a constant speed c.

That is, the image at time t is given by:

$$I^*(x,y) = I(x, y-ct)$$

Then: the degradation function is given by:

$$H(u,v) = \int_0^T e^{-j\frac{2\pi}{N}(v(ct))}\,dt$$

Remark: Once the degradation function is known, the original image can be restored by: $\mathrm{IDFT}\left(\frac{G(u,v)}{H(u,v)}\right)$ (given that there's no noise)

What if there is noise??

Image deblurring in the frequency domain: (Assume H is known)

In the frequency domain,

$$\boxed{\underbrace{G(u,v)}_{\mathrm{DFT}(g)} = cH(u,v)\,\underbrace{F(u,v)}_{=\,\mathrm{DFT}(f)}} + \underbrace{N(u,v)}_{=\,\mathrm{DFT}(n)}$$

$$\frac{G(u,v)}{cH(u,v)} \approx \cancel{F(u,v)} = \frac{cH(u,v)F(u,v) + N(u,v)}{cH(u,v)}$$

$$= F(u,v) + \frac{N(u,v)}{cH(u,v)}$$

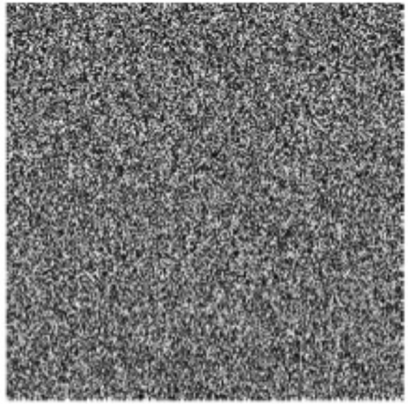
$H(u,v)$ blurs the image

LPF = $H(u,v)$ is big $(u,v) \approx (0,0)$

$H(u,v)$ is small (u,v) far away from $(0,0)$

Original

Blurred image



Direct inverse filtering