Lecture 5:

Recap:

Haar transformation

Definition: (Haar functions) The Haar functions are defined recursively as follows

$$H_0(t) \equiv \begin{cases} 1 & \text{if } 0 \le t < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

$$H_1(t) \equiv \begin{cases} 1 & \text{if } 0 \le t < \frac{1}{2} \\ -1 & \text{if } \frac{1}{2} \le t < 1 \\ 0 & \text{elsewhere} \end{cases}$$
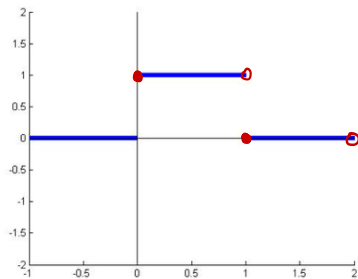
$$H_{2^p+n} \equiv \begin{cases} \sqrt{2}^{\,p} & \text{if } \frac{n}{2^p} \le t < \frac{n+0.5}{2^p} \\ -\sqrt{2}^{\,p} & \text{if } \frac{n+0.5}{2^p} \le t < \frac{n+1}{2^p} \\ 0 & \text{elsewhere} \end{cases}$$
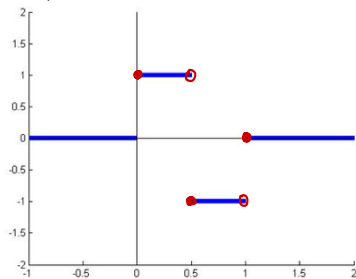
where $p = 1, 2, \dots;\ n = 0, 1, 2, \dots, 2^p - 1$

Remark: If p is larger, H_{2^p+n} is compactly supported in a smaller region.
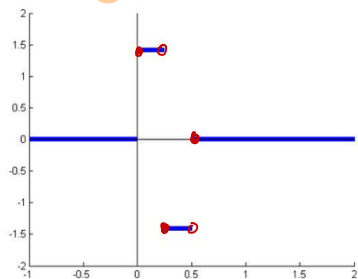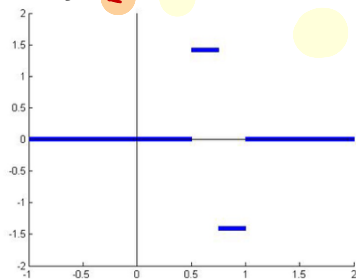
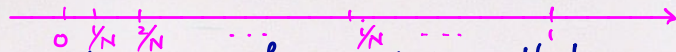# Examples of Haar functions:

H_0

H_1

$H_2 = 2^1 + 0$

$H_3 = 2^4 + 1$

= p

= n

Definition (Discrete Haar Transform)

The Haar Transform of a $N \times N$ image is done by dividing $[0,1]$ into partitions.

Let $H(k,i) \equiv H_k\left(\frac{i}{N}\right)$ where $k, i = 0, 1, 2, \ldots, N-1$.
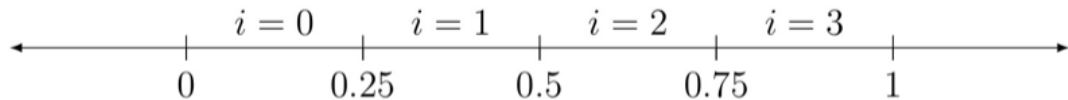
We obtain the Haar Transform matrix: $\widetilde{H} \equiv \frac{1}{\sqrt{N}} H$ where $H \equiv \left(H(k,i)\right)_{0 \leq k, i \leq N-1}$

The Haar Transform of $f \in M_{n \times n}$ is defined as:

$$g = \widetilde{H} f \widetilde{H}^T.$$

Example   Compute the Haar Transform matrix for a 4×4 image.

Solution: Divide $[0, 1]$ into 4 portions:

Need to check:

H_0
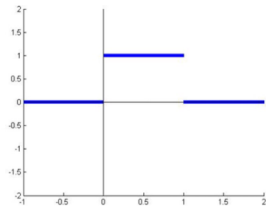
H_1
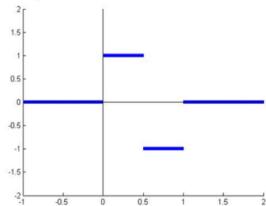
H_2
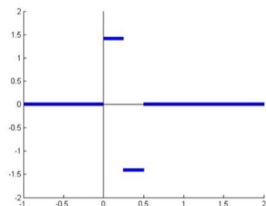
H_3
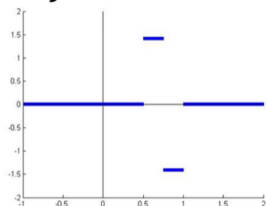
We get that:

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \quad \text{and} \quad \tilde{H} = \frac{1}{\sqrt{4}}H = \frac{1}{2}H$$

Easy to check that $\tilde{H}^T \tilde{H} = I$.

Elementary images under Haar transform:

Using Haar transform, f can be written as:

$$f = \tilde{H}^T g \tilde{H}$$

$\underset{\text{transformed image}}{\longleftarrow}$

Let $\tilde{H} = \begin{pmatrix} - & \vec{h}_1^T & - \\ - & \vec{h}_2^T & - \\ & \vdots & \\ - & \vec{h}_N^T & - \end{pmatrix}$. Then: $f = \sum_{i=1}^{N} \sum_{j=1}^{N} g_{ij} \underbrace{\vec{h}_i \vec{h}_j^T}_{I_{ij}^H}$

$I_{ij}^T = $ elementary images under Haar Transform.

<u>Definition: (Walsh function)</u> The Walsh functions are defined recursively by:

$$W_{2j+q}(t) \equiv (-1)^{\lfloor \frac{j}{2} \rfloor + q} \left\{ W_j(2t) + (-1)^{j+q} W_j(2t-1) \right\}$$

where $\lfloor \frac{j}{2} \rfloor$ = biggest integer smaller than or equal to $\frac{j}{2}$.

$q = 0$ or 1, $j = 0, 1, 2, \ldots$ and

$$W_0(t) \equiv \begin{cases} 1 & 0 \leq t < 1 \\ 0 & \text{elsewhere} \end{cases}$$
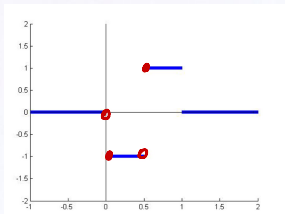
<u>Example:</u> Compute $W_1(t)$.

Put $j = 0$, $q = 1$. Then:

$$W_1(t) = (-1)^{\lfloor 0 \rfloor + 1} \left\{ W_0(2t) + (-1)^1 W_0(2t-1) \right\} = (-1) \left\{ W_0(2t) + (-1)^1 W_0(2t-1) \right\}$$

For $0 \leq t < \frac{1}{2}$, $W_0(2t) = 1$, $W_0(2t-1) = 0$ $\Rightarrow W_1(t) = -1$.

For $\frac{1}{2} \leq t < 1$, $W_0(2t) = 0$, $W_0(2t-1) = 1$ $\Rightarrow W_1(t) = 1$.

Definition: (Discrete Walsh transform)

The Walsh Transform of a $N \times N$ image is defined as follows.

Define $W(k,i) \equiv W_k\left(\frac{i}{N}\right)$ where $k, i = 0, 1, 2, \ldots, N-1$.

The Walsh transform matrix is:

$$\tilde{W} \equiv \frac{1}{\sqrt{N}} W \quad \text{where} \quad W \equiv \left(W(k,i)\right)_{0 \le k, i \le N-1}$$

The Walsh transform of $f \in M_{n \times n}$ is defined as:

$$g = \tilde{W} f \tilde{W}^T$$

Example   Compute the Walsh Transform matrix for a 4×4 image.

Solution: Again, divide $[0, 1]$ into 4 portions:
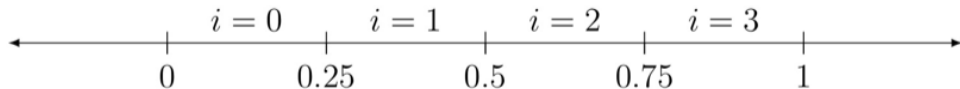
We can check that:

W_0
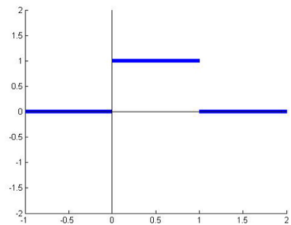
W_1
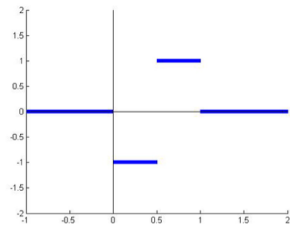
W_2
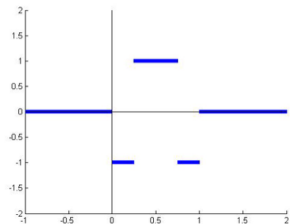
W_3
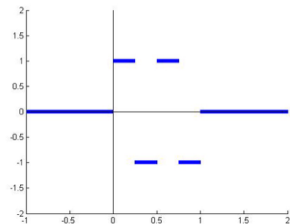
So,

$$W = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \quad \text{and} \quad \tilde{W} = \frac{1}{\sqrt{4}}W = \frac{1}{2}W$$

$(\tilde{W}^T \tilde{W} = I)$

Example 2.7: Compute the Walsh Transform of

$$f = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Solution:

$$g = \tilde{W} f \tilde{W}^T = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Big\}$ More zeros in the coefficient matrix!

Remark: 1. Walsh transform is to transform an image to a "transformed image" with much more zeros.

Elementary images under Walsh transform:

Under Walsh Transform, $f = \widetilde{W}^T g \widetilde{W}$.

transformed image

Then: $f = \sum\limits_{i=1}^{N} \sum\limits_{j=1}^{N} g_{ij} \underbrace{\vec{W_i} \vec{W_j}^T}_{\text{''} I_{ij}^W}$ where $\widetilde{W} = \begin{pmatrix} - & \vec{W_1}^T & - \\ - & \vec{W_2}^T & - \\ & \vdots & \\ - & \vec{W_N}^T & - \end{pmatrix}$

I_{ij}^W = elementary images under Walsh transform.
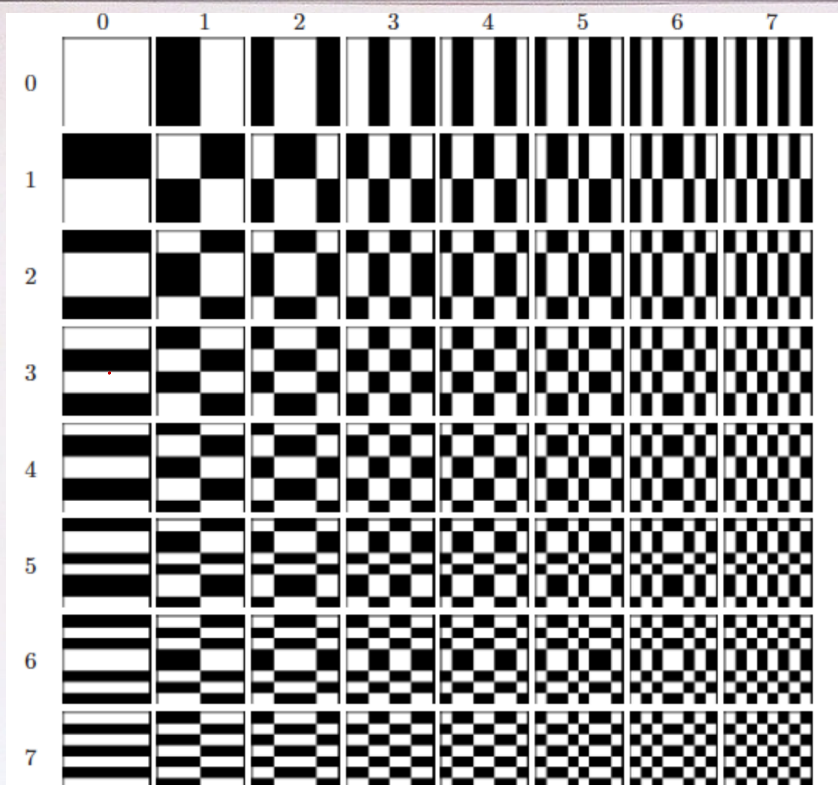
Discrete Fourier Transform:

Definition: The 1D discrete Fourier Transform (DFT) of a function $f(k)$, defined at discrete points $k = 0, 1, 2, \ldots, N-1$ is defined as:

$$\hat{f}(m) = \frac{1}{N} \sum_{k=0}^{N-1} f(k) \, e^{-j \frac{2\pi mk}{N}} \quad \left(\text{where } j = \sqrt{-1}, \; e^{j\theta} = \cos\theta + j\sin\theta \right)$$

The 2D DFT of a $M \times N$ image $g = (g(k, l))_{k, l}$, where $0 \leq k \leq M-1$, $0 \leq l \leq N-1$ is defined as:

$$\hat{g}(m, n) = \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} g(k, l) \, e^{-j 2\pi \left(\frac{km}{M} + \frac{ln}{N} \right)}$$

Remark: The inverse of DFT is given by:

$$g(p, q) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \underbrace{\hat{g}(m, n)}_{\text{DFT of } g} \; e^{j 2\pi \left(\frac{pm}{M} + \frac{qn}{N} \right)}$$

$\left(\text{no } \frac{1}{Mn}! \right)$

$(\text{no } -\text{ve sign})$

Proof of Inverse DFT:

$$\sum_{m=0}^{M-1} \sum_{n=0}^{N-1} e^{j2\pi\left(\frac{pm}{M} + \frac{qn}{N}\right)} \hat{g}(m,n) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} e^{j2\pi\left(\frac{pm}{M} + \frac{qn}{N}\right)} \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} g(k,l) \, e^{-j2\pi\left(\frac{km}{M} + \frac{ln}{N}\right)}$$

$$= \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} g(k,l) \, e^{j2\pi\left(\frac{(p-k)m}{M} + \frac{(q-l)n}{N}\right)}$$

$$= \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} g(k,l) \underbrace{\sum_{m=0}^{M-1} e^{j2\pi\left(\frac{(p-k)m}{M}\right)} \sum_{n=0}^{N-1} e^{j2\pi\left(\frac{(q-l)n}{N}\right)}}_{(*)}$$

Note that:
$$\sum_{m=0}^{M-1} e^{j2\pi\left(\frac{mt}{M}\right)} = \frac{\left[e^{j2\pi\left(\frac{t}{M}\right)}\right]^M - 1}{\underbrace{e^{j2\pi\left(\frac{t}{M}\right)} - 1}_{\text{if } t \neq 0}} = M\,\delta(t) := \begin{cases} M & t = 0 \\ 0 & t \neq 0 \end{cases}$$

$\therefore$ (*) becomes:
$$\frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} g(k,l) \, M\,\delta(p-k) \, N\,\delta(q-l) = g(p,q).$$

Image decomposition under DFT:

Consider a $N \times N$ image g, the DFT of g:

$$\hat{g}(m,n) = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} g(k,l) \, e^{-j2\pi\left(\frac{km+ln}{N}\right)}$$

Define $\mathcal{U}_{kl} = \frac{1}{N} e^{-j\frac{2\pi kl}{N}}$ where $0 \leq k, l \leq N-1$ and $\mathcal{U} = (\mathcal{U}_{kl})_{0 \leq k, l \leq N-1} \in M_{N \times N}$

$\mathcal{U}$ is clearly symmetric and also:

$$\hat{g} = \mathcal{U} g \mathcal{U} \quad \text{(DFT in matrix form)}$$

Note that: $\frac{1}{N^2} \sum_{\alpha=0}^{N-1} e^{-j\left(\frac{2\pi x_1 \alpha}{N}\right)} e^{+j\left(\frac{2\pi x_2 \alpha}{N}\right)} = \frac{1}{N^2} \sum_{\alpha=0}^{N-1} e^{-j\frac{2\pi(x_2-x_1)\alpha}{N}} = \frac{1}{N^2} N \delta(x_2 - x_1)$

$$= \frac{1}{N} \delta(x_2 - x_1)$$

Let $\mathcal{U} = \left(\frac{1}{N} \vec{u}_1 \; \frac{1}{N} \vec{u}_2 \ldots \frac{1}{N} \vec{u}_N \right)$. Then: $\langle \vec{u}_i, \vec{u}_j \rangle = \vec{u}_i^T \overline{\vec{u}_j} = \begin{cases} \frac{1}{N} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

$\therefore \{\vec{u}_i\}_{i=1}^N$ is orthogonal but NOT orthonormal!

$$\hat{g}(m,n) = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} g(k,l) \, e^{-j2\pi\left(\frac{km+ln}{N}\right)}$$

$$\sum_{k=0}^{N-1} \underbrace{\frac{e^{-j2\pi\frac{km}{N}}}{N}}_{U_{mk}} \underbrace{\sum_{l=0}^{N-1} g(k,l) \left(\underbrace{\frac{e^{-j\frac{2\pi ln}{N}}}{N}}_{} \right)}_{U_{ln}}$$

$$\underbrace{\qquad\qquad}_{gU(k,n)}$$

$$U(gU)(m,n)$$

$$\therefore \boxed{\hat{g} = U g U}$$

$$\therefore \; UU^* = \frac{1}{N} I = U^* U$$

$$\therefore \; g = (NU)^* \, \hat{g} \, (NU)^*$$

$$\therefore \; g = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \hat{g}_{kl} \, \boxed{\vec{W}_k \, \vec{W}_l^{\,T}} \longleftarrow \text{Elementary image of DFT}$$

where $\vec{W}_k = k^{th}$ col of $(NU)^*$

$$\hat{g} = U \, g \, U$$

$$\Rightarrow \; U^* \, \hat{g} \, U^* = (U^* U) \, g \, (U U^*)$$

$$= \left(\frac{1}{N}\right) g \left(\frac{1}{N}\right)$$

$$\therefore \; (NU)^* \, \hat{g} \, (NU)^* = g \; //$$

Example   Find the DFT of the following 4×4 image

$$g = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Solution

The matrix U is given by:

$$U = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix}$$

$$\therefore \text{DFT of } g = \hat{g} = UgU = \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathcal{U} = \left(\; \mathcal{U}_{k\ell} \; \right)_{k,\ell}$$

$$= \frac{1}{4}\left(e^{-j 2\pi \left(\frac{k\ell}{4} \right)} \right)$$

Why is DFT useful in imaging:

1. DFT of convolution:

   Recall: $g * w (n, m) = \sum\limits_{n'=0}^{N-1} \sum\limits_{m'=0}^{N-1} g(n-n', m-m') \, w(n', m')$

   $\left(g, m \in M_{N \times M}(\mathbb{R}) \right)$

Then, the DFT of $g * w$ = $MN \, DFT(g) \, DFT(w)$

∴ DFT of convolution can be reduced to simple multiplication!

Recall: Shift-invariant image transformation = 2D convolution.

∴ Easy computation/manipulation of shift-invariant transf.
after DFT!!

Proof:

DFT of $g*\omega$ at (p, q)

$= \dfrac{1}{NM} \sum\limits_{n=0}^{N-1} \sum\limits_{m=0}^{M-1} g*\omega(n,m) \, e^{-j2\pi\left(\frac{pn}{N} + \frac{qm}{M}\right)}$

$= \dfrac{1}{NM} \sum\limits_{n'=0}^{N-1} \sum\limits_{m'=0}^{M-1} \sum\limits_{n=0}^{N-1} \sum\limits_{m=0}^{M-1} g(n-n', m-m') \, \omega(n',m') \, e^{-j2\pi\left(\frac{pn}{N} + \frac{qm}{M}\right)}$

Change of variables:

$n \to n'' = n - n'$
$m \to m'' = m - m'$

$= \dfrac{1}{NM} \sum\limits_{n'=0}^{N-1} \sum\limits_{m'=0}^{M-1} \omega(n',m') \, e^{-j2\pi\left(\frac{pn'}{N} + \frac{qm'}{M}\right)} \underbrace{\sum\limits_{n''=-n'}^{N-1-n'} \sum\limits_{m''=-m'}^{M-1-m'} g(n'',m'') \, e^{-j2\pi\left(\frac{pn''}{N} + \frac{qm''}{M}\right)}}$

$\underbrace{\hspace{5cm}}_{\hat{\omega}(p,q)}$

$\underbrace{\hspace{5cm}}_{T(p,q)}$

Note that: g and ω are periodically extended.

$\therefore \; g(n-N, m) = g(n,m)$ and $g(n, m-M) = g(n,m)$

$\therefore \; T = \sum\limits_{m''=-m'}^{M-1-m'} e^{-j2\pi \frac{qm''}{M}} \sum\limits_{n''=-n'}^{-1} g(n'',m'') \, e^{-j2\pi\frac{pn''}{N}} + \sum\limits_{m''=-m'}^{M-1-m'} e^{-j2\pi\frac{qm''}{M}} \sum\limits_{n''=0}^{N-1-n'} g(n'',m'') \, e^{-j2\pi\left(\frac{pn''}{N}\right)}$

Consider $\displaystyle\sum_{n''=-n'}^{-1} g(n'', m'')\, e^{-j\,2\pi \frac{p\,n''}{N}}$ $\boxed{n'''=N+n''}$ $\displaystyle\sum_{n'''=N-n'}^{N-1} g(n'''-N,\, m'')\, e^{-j\,2\pi\left(\frac{p\,n'''}{N}\right)} e^{j\,2\pi p}$

$\underbrace{\qquad}_{g(n''',\,m'')}$

$e^{-j\,2\pi\left(\frac{p\,n''}{N}\right)} = n'''-N$

We can do similar thing for index m''.

$\therefore\ T = \displaystyle\sum_{m''=0}^{M-1}\sum_{n''=0}^{N-1} g(n'', m'')\, e^{-j\,2\pi\left(\frac{p\,n''}{N} + \frac{q\,m''}{M}\right)} = MN\,\hat{g}(p, q)$

$\therefore\ \widehat{g * w}\,(p, q) = MN\,\hat{g}(p, q)\,\hat{w}(p, q)$

Remark: Conversely, if $x(n, m) = g(n, m)\, w(n, m)$

Then, $\hat{x}(k, l) = \displaystyle\sum_{p=0}^{N-1}\sum_{q=0}^{M-1} \hat{g}(p, q)\,\hat{w}(k-p, l-q)$ (Convolution of g and w)