Lecture 3

Recap: Error between images : $\|f - g\|$. Need to define a norm.

Another commonly used matrix norm

Definition: (Frobenius norm) $$\|A\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2}$$

Let $\vec{a}_j$ = j-th col of A. We have: $\|A\|_F = \sqrt{\sum_{j=1}^{n} \|a_j\|_2^2} = \sqrt{tr(A^*A)} = \sqrt{tr(AA^*)}$

where $tr(\cdot)$ = trace of the matrix.

Theorem: The Frobenius norm (F-norm) is invariant under multiplication by orthogonal matrices.

That is, for any $A \in \mathbb{R}^{m\times n}$, and orthogonal matrix $U \in \mathbb{R}^{m\times m}$, we have:

$$\|UA\|_F = \|A\|_F.$$

Proof:

$$\|UA\|_F = \sqrt{tr((UA)^T(UA))} = \sqrt{tr(A^T U^T U A)} = \sqrt{tr(A^T A)} = \|A\|_F$$

Recap:

Image decomposition

Suppose $h(x,\alpha,y,\beta) = h_c(x,\alpha)\, h_r(y,\beta)$ (Separable).

Then: $g = h_c^T f h_r \Rightarrow f = (h_c^T)^{-1} g (h_r)^{-1}$

Write: $(h_c^T)^{-1} = \begin{pmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_N \\ | & | & & | \end{pmatrix}$; $h_r^{-1} = \begin{pmatrix} - & \vec{v}_1^T & - \\ - & \vec{v}_2^T & - \\ & \vdots & \\ - & \vec{v}_N^T & - \end{pmatrix}$

Then: $f = \sum_{i=1}^{N} \sum_{j=1}^{N} g_{ij} \, \vec{u}_i \vec{v}_j^T$ $M_{N\times N}$

Check that: $(h_c^T)^{-1} \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & 1 & \vdots \\ 0 & \cdots & 0 \end{pmatrix} h_r^{-1} = \vec{u}_i \vec{v}_j^T$

(i,j)-entry

$\therefore f$ = linear combination of $\{\vec{u}_i \vec{v}_j^T\}_{i,j}$

Image decomposition

Image decomposition based on Singular Value Decomposition (SVD)

Definition: (SVD) For any $g \in \mathbb{R}^{m \times n}$, the singular value decomposition (SVD) of g is a matrix factorization: $g = U \Sigma V^T$, where $U \in \mathbb{R}^{m \times m}$, $V \in \mathbb{R}^{n \times n}$ are unitary, Σ is a diagonal matrix ($\Sigma_{ij} = 0$ if $i \neq j$) with diagonal entries given by: $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0$ with $r \leq \min(m, n)$.

Theorem: The rank of g is given by the number of non-zero singular values.

Proof: Rank = dim of column space.

Recall that $\text{rank}(AB) = \text{rank}(B)$ if A is invertible

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Suppose $g = U \Sigma V^T$. Since U and V are invertible, $\text{rank}(g) = \text{rank}(\Sigma)$

$= \#$ of non-zero singular values

Theorem: $\text{Range}(g) = \text{span}\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_r\}$ where $U = \begin{pmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \dots & \vec{u}_m \\ | & | & & | \end{pmatrix}$; $r = \text{rank}(g)$

$\text{Null}(g) = \text{span}\{\vec{v}_{r+1}, \dots, \vec{v}_n\}$ where $V = \begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \\ | & | & & | \end{pmatrix}$.

Proof: Exercise.

Theorem: (Existence of SVD) Every $m \times n$ image has a SVD.

Proof: Later!

$$\text{Null}(g) = \{\vec{x} : g\vec{x} = \vec{0}\}$$

How to compute SVD

Let $A \in M_{m\times n}$ $(m \geq n)$ $(\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n)$

Step 1: Find eigenvalues $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ and orthonormal eigenvectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} \rightarrow V$

symmetric of $A^TA \in M_{n\times n}$ (with $\|\vec{v}_j\| = 1$, $j = 1, \dots, n$)

[Recall: $(A^TA)\vec{v}_j = \lambda_j \vec{v}_j$]

Step 2: Define:

$$\underline{\underline{\Sigma}} = \begin{pmatrix} \overset{=\sigma_1}{\sqrt{\lambda_1}} & & & \\ & \overset{=\sigma_2}{\sqrt{\lambda_2}} & & \\ & & \ddots & \\ & & & \overset{=\sigma_n}{\sqrt{\lambda_n}} \\ \hline & & 0 & \end{pmatrix} \in M_{m\times n}$$

Add zero rows if $m > n$

Step 3: For non-zero $\sigma_1, \sigma_2, \dots, \sigma_r$,

let $\vec{u}_1 = \dfrac{A\vec{v}_1}{\sigma_1}$, $\vec{u}_2 = \dfrac{A\vec{v}_2}{\sigma_2}$, ..., $\vec{u}_r = \dfrac{A\vec{v}_r}{\sigma_r}$

Step 4: Extend $\{\vec{u}_1, \dots, \vec{u}_r\}$ to the basis $\{\vec{u}_1, \dots, \vec{u}_r, \dots, \vec{u}_m\}$ of $\mathbb{R}^m$. $\rightarrow U$

Step 5: Let:

$$U = \begin{pmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_m \\ | & | & & | \end{pmatrix} \in M_{m\times m}$$

$$V = \begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{pmatrix} \in M_{n\times n}$$

Then: $A = U\Sigma V^T$

Example 2.1: Let

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{pmatrix} .$$

We have

$$A^T A = \begin{pmatrix} 9 & 8 \\ 8 & 9 \end{pmatrix} . \qquad \text{(Step 1)}$$

Now, eig($A^T A$) are 17 and 1, and so $\sigma_1 = \sqrt{17}$, $\sigma_2 = 1$ and

$$\Sigma = \begin{pmatrix} \sqrt{17} & 0 \\ 0 & 1 \end{pmatrix} .$$

Moreover,

$$\vec{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} , \ \vec{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} .$$

This gives

$$V = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}$$

$$u_i = \frac{A\vec{v}_i}{\sigma_i}$$

Since

$$\sigma_1 \vec{u}_1 = A\vec{v}_1 ,$$

we have

$$\vec{u}_1 = \underbrace{\frac{1}{\sqrt{17}}}_{\sigma_1} \frac{1}{\sqrt{2}} \overbrace{\begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{pmatrix}}^{A} \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\vec{v}_1} = \frac{1}{\sqrt{34}} \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix} .$$

$$\frac{A\vec{v}_2}{\sigma_2}$$

$$\parallel$$

Similarly, we have

$$\vec{u}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

The matrix U is, therefore, given by

$$U = \begin{pmatrix} \frac{3}{\sqrt{34}} & \frac{-1}{\sqrt{2}} & \\ \frac{4}{\sqrt{34}} & 0 & \mathbf{u}_3 \\ \frac{3}{\sqrt{34}} & \frac{1}{\sqrt{2}} & \end{pmatrix}$$

for some vector $\mathbf{u}_3$ orthonormal to both $\mathbf{u}_1$ and $\mathbf{u}_2$. One possibility is

$$\vec{u}_3 = \frac{1}{\sqrt{17}} \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} .$$

Finally, the SVD of A is given by

$$\begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{34}} & \frac{-1}{\sqrt{2}} & \frac{2}{\sqrt{17}} \\ \frac{4}{\sqrt{34}} & 0 & \frac{-3}{\sqrt{17}} \\ \frac{3}{\sqrt{34}} & \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{17}} \end{pmatrix} \begin{pmatrix} \sqrt{17} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} .$$

Remark:

1. Note that $gg^T = U\Lambda^{1/2}\underbrace{V^TV}_{I}\Lambda^{1/2}U^T = U\Lambda U^T$

   $\therefore$ U consists of eigenvectors of gg^T.

   Note that $g^Tg = V\Lambda^{1/2}\underbrace{U^TU}_{I}\Lambda^{1/2}V^T = V\Lambda V^T$

   $\therefore$ V consists of eigenvectors of g^Tg.

2. Note that
$$g = U\Lambda^{1/2}V^T = \sum_{i=1}^{r}\sigma_i\, U\begin{array}{c} \quad\quad i\text{th} \\ \quad\quad\downarrow \\ \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix} \leftarrow i\text{th} \end{array} V^T = \sum_{i=1}^{r}\sigma_i\,\vec{u}_i\,\vec{v}_i^{\,T}$$

   $\vec{u}_i\vec{v}_i^{\,T}$ is called the eigen-image of g under SVD.

3. For $N\times N$ image, the required storage is:
$$(\underbrace{N}_{\vec{u}_i} + \underbrace{N}_{\vec{v}_i} + \underbrace{1}_{\sigma_i})\times\underbrace{r}_{r\text{ terms}} = (2N+1)r$$

Definition: For any k $(0\le k\le r)$, we define
$$g_k = \sum_{j=1}^{k}\sigma_j\,\vec{u}_j\,\vec{v}_j^{\,T} \quad (\text{rank-}k\text{ approximation of } g)$$