MMAT5390: Mathematical Image Processing

Assignment 3

April 1 2022

Please give reasons in your solutions.

1. Suppose the definition of the DFT on $N \times N$ images is changed to

$$\hat{f}(m,n) = DFT(f)(m,n) = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} f(k,l)e^{2\pi j\frac{mk+nl}{N}}.$$

   (a) Does there exist a matrix U such that $\hat{f} = UfU$ for an $N \times N$ image f? If yes, derive U and check if it is unitary.

   (b) Show that the inverse DFT (iDFT) is defined by

$$f(p,q) = iDFT(\hat{f})(p,q) = \frac{1}{N}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1} \hat{f}(m,n)e^{-2\pi j\frac{pm+qn}{N}}.$$

2. Suppose $\hat{f}$ and $\hat{g}$ are the Fourier coefficients of DFT of $M \times N$ images f and g. Prove the convolution theorem that $\hat{f} * \hat{g} = \widehat{f \odot g}$, where $f \odot g(k,l) = f(k,l)g(k,l)$.

3. The definition of even discrete cosine transform (EDCT) on $N \times N$ images is as follow

$$\hat{f}(m,n) = EDCT(f)(m,n) = \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} f(k,l)\cos\frac{\pi m(k+\frac{1}{2})}{N}\cos\frac{\pi n(l+\frac{1}{2})}{N}.$$

   (a) Suppose f is an $N \times N$ image, derive the matrix T such that $\hat{f} = TfT^T$.

   (b) Suppose $N > 1$. Prove that for any $c \in \mathbb{R}$, cT is not unitary.

4. Consider a Gaussian low-pass filter

$$H(u,v) = exp\left(-\frac{u^2+v^2}{2\sigma^2}\right).$$

   Suppose $H(4,2) = \frac{1}{\sqrt{e}}H(1,-3)$. Find σ^2.

5. Given an image $I = (I(m,n))$, where $-N \leq m,n \leq N-1$.

   The Butterworth low-pass filter H, with radius D_0 and order n, is applied on the Fourier coefficients $DFT(I) = (\hat{I}(u,v))$ to obtain $G(u,v)$, where $-N \leq u,v \leq N-1$. The filter H is

$$H(u,v) = \frac{1}{1+(D(u,v)/D_0^2)^n}$$

   Suppose it satisfies

$$\begin{cases} \hat{I}(1,0) \neq 0 \\ \hat{I}(-1,1) \neq 0 \\ G(1,0) = \frac{64}{65}\hat{I}(1,0) \\ G(-1,1) = \frac{8}{9}\hat{I}(-1,1) \end{cases}$$

   Find D_0 and n.

6. **Coding assignment**: Please read the MATLAB file **or** the Jupyter notebook file in the attached zip file carefully. There are missing lines in the file. You can either choose MATLAB or Python to finish. Add the missing lines by yourself and test the file using the given image. (Note: In this coding assignment, we discuss the image processing of grayscale images only.)

   Coding instruction: In assignment 2, we implement DFT hand by hand. In this assignment, we use the built-in function ($fft2(ifft2)$ and $fftshift(ifftshift)$ in MATLAB or $numpy.fft$ in Python) to simplify our code. The only difference between $fft2$ (or $numpy.fft.fft2$) and our definition of DFT is that our definition is $\frac{1}{N^2}$ of $fft2$. Similarly, Our definition of iDFT is N^2 of $ifft2$.

   The built-in function $fftshift$ and $ifftshift$ in MATLAB or the corresponding implementations in $numpy.fft$ module have the same effect as $circshift(x, [h/2, w/2])$, which shifts frequency components at the corner to the center. So we use the two functions to replace $circshift$ to simplify the code.

   In order to compute the distance between each element and the center, we make use of the built-in function $meshgrid$ in MATLAB or $numpy.meshgrid$ in Python. Here is an example of the function $[X, Y] = meshgrid([-2:2], [-1:1])$:

$$X = \begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ -2 & -1 & 0 & 1 & 2 \\ -2 & -1 & 0 & 1 & 2 \end{pmatrix} \qquad Y = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

   You are required to define the Gaussian Low Pass Filter using the definition

$$H(u, v) = e^{-\frac{u^2+v^2}{2\sigma^2}}$$

   where $-\frac{N}{2} \le u \le \frac{N}{2} - 1$ and $-\frac{N}{2} \le v \le \frac{N}{2} - 1$.
   The results are as follows

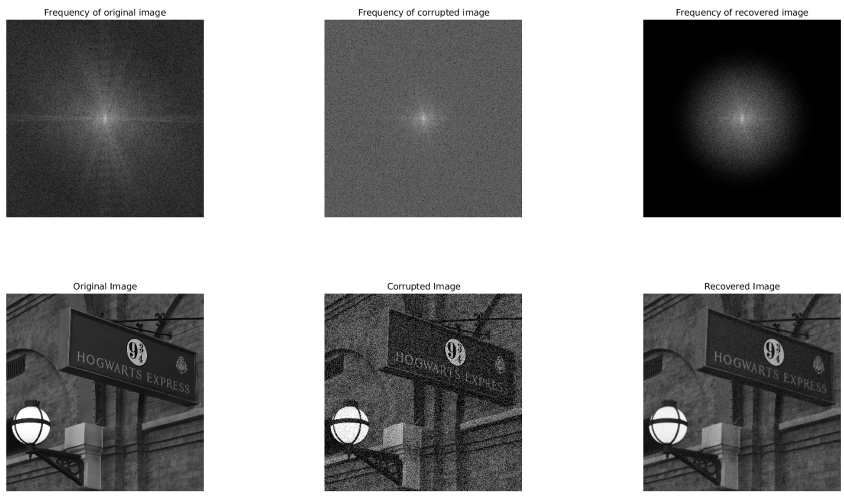

Figure 1: Experimental Results