MMAT5390: Mathematical Image Processing

Assignment 2

Due: March 8 2022

Please give reasons in your solutions.

1. Let $H_n(t)$ denotes the n^{th} Haar function, where $n \in \mathbb{N} \cup \{0\}$.

   (a) Give the definition of $H_n(t)$, and derive the Haar transform matrix for 4×4 images according to the definition.

   (b) Let $A = \begin{pmatrix} 5 & 3 & 9 & 0 \\ 3 & 9 & 0 & 5 \\ 9 & 0 & 5 & 3 \\ 0 & 5 & 3 & 9 \end{pmatrix}$. Using the Haar transform matrix derived above to compute the Haar transform A_{Haar} of A.

   (c) By setting the four smallest (in absolute value) nonzero elements of A_{Haar} to 0, we obtain $\tilde{A}_{\text{Haar}}$. Compute the reconstructed image $\tilde{A}$ of $\tilde{A}_{\text{Haar}}$.

2. Let $W_n(t)$ denotes the n^{th} Walsh function, where $n \in \mathbb{N} \cup \{0\}$.

   (a) Give the definition of $W_n(t)$, and derive the Walsh transform matrix for 4×4 images according to the definition.

   (b) Let $B = \begin{pmatrix} 1 & 3 & 7 & 8 \\ 3 & 7 & 6 & 2 \\ 2 & 6 & 3 & 5 \\ 2 & 8 & 4 & 1 \end{pmatrix}$. Using the Walsh transform matrix derived above to compute the Walsh transform B_{Walsh} of B.

   (c) By setting the largest element of B_{Walsh} to 1, we obtain $\tilde{B}_{\text{Walsh}}$. Compute the reconstructed image $\tilde{B}$ of $\tilde{B}_{\text{Walsh}}$.

3. Let $H_n(t)$ denotes the n^{th} Haar function, where $n \in \mathbb{N} \cup \{0\}$. We denote the inner product of two functions f and g as

$$L^2(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{R} \,\middle|\, \int_{\mathbb{R}} f^2 < \infty \right\},$$

   and for any $f, g \in L^2(\mathbb{R})$,

$$\langle f, g \rangle = \int_{\mathbb{R}} fg.$$

   (a) (Unit) Prove that $\int_{\mathbb{R}} [H_m(t)]^2 \, dt = 1$ for any $m \in \mathbb{N} \cup \{0\}$.
   (Hence $H_m \in L^2(\mathbb{R})$ and $\|H_m\| = 1$.)

   (b) (Orthogonality)

      i. Prove that $\langle H_0, H_m \rangle = 0$ for any $m \in \mathbb{N} \setminus \{0\}$.

      ii. Let $m_1, m_2 \in \mathbb{N}$ such that $0 \neq m_1 < m_2$. Then $m_1 = 2^{p_1} + n_1$ and $m_2 = 2^{p_2} + n_2$ for some $p_1, p_2 \in \mathbb{N} \cup \{0\}$, $n_1 \in \mathbb{Z} \cap [0, 2^{p_1} - 1]$ and $n_2 \in \mathbb{Z} \cap [0, 2^{p_2} - 1]$.

         A. Suppose $p_1 = p_2$. Prove that $\langle H_{m_1}, H_{m_2} \rangle = 0$.
         Hint. In this case $n_1 < n_2$.

         B. Suppose $p_1 < p_2$.

         - Show that the length of $\left[0, \frac{n_1}{2^{p_1}}\right)$ is a multiple of that of $\left[\frac{n_2}{2^{p_2}}, \frac{n_2+1}{2^{p_2}}\right)$.
         - Show that the length of $\left[\frac{n_1}{2^{p_1}}, \frac{n_1+0.5}{2^{p_1}}\right)$ is a multiple of that of $\left[\frac{n_2}{2^{p_2}}, \frac{n_2+1}{2^{p_2}}\right)$.

- Show that the length of $\left[\frac{n_1+0.5}{2^{p_1}}, \frac{n_1+1}{2^{p_1}}\right)$ is a multiple of that of $\left[\frac{n_2}{2^{p_2}}, \frac{n_2+1}{2^{p_2}}\right)$.
- According to the above steps, considering the possible subset relations between the supports of H_{m_1} and H_{m_2}, prove that $\langle H_{m_1}, H_{m_2}\rangle = 0$.

The above establishes that $\mathcal{H}$ is orthonormal in $(L^2(\mathbb{R}), \langle\cdot,\cdot,\rangle)$.

4. (a) Give the definition of 2D discrete Fourier transform of an $M \times N$ image, and write down the Fourier transform matrix U for 4×4 images.

   (b) Let $C = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}$. Using the Fourier transform matrix U derived above to compute the Fourier transform C_{DFT} of C.

   (c) By setting the smallest (in modulus value) nonzero elements of C_{DFT} to 0, we obtain $\tilde{C}_{\text{DFT}}$. Compute the reconstructed image $\tilde{C}$ of $\tilde{C}_{\text{DFT}}$.

5. **Coding assignment**: Please read the MATLAB file **or** the Jupyter notebook file in the attached zip file carefully. There are missing lines in the file. You can either choose MATLAB or Python to finish. Add the missing lines by yourself and test the file using the given image. (Note: In this coding assignment, we discuss the image processing of grayscale images only.)

   Coding instruction:

   Q1: Recall that DFT can be rewritten as matrix multiplication.

   $$\hat{g} = UgU \tag{1}$$

   where $U_{\alpha\beta} = \frac{1}{N} e^{-2\pi j \frac{\alpha\beta}{N}}$ where $0 \le \alpha, \beta \le N-1$, and $U = (U_{\alpha\beta})_{0\le\alpha,\beta\le N-1} \in M_{N\times N}(\mathbb{C})$.

   In this coding assignment, you are required to reconstruct the image given a modified $\hat{g}$, which represents the Fourier coefficients. You are **not allowed** to use the built-in MATLAB function $ifft2$ or any Python Fourier transform module such as numpy.fft module and scipy.fft.

   Later in this course, we will do image processing in the spectral domain. We will use this technique again and again.