MMAT5390: Mathematical Image Processing

Assignment 4

1. (a) D is block-circulant, i.e.

$$D = \begin{pmatrix} D_0 & D_{N-1} & D_{N-2} & \cdots & D_1 \\ D_1 & D_0 & D_{N-1} & \cdots & D_2 \\ D_2 & D_1 & D_0 & \cdots & D_{2,N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ D_{N-1} & D_{N-2} & D_{N-3} & \cdots & D_0 \end{pmatrix},$$

with each $N \times N$ block $D_0, D_1, \cdots, D_{N-1}$ being circulant. Hence denoting the (k,l)-th block of a $N \times N$ matrix A of $N \times N$ blocks by $A_{k,l}$, we have:

$$\begin{aligned}
(W^{-1}DW)_{k,l} &= \sum_{m=0}^{N-1} W^{-1}_{k,m}(DW)_{m,l} \\
&= \sum_{m=0}^{N-1} W^{-1}_{k,m} \sum_{n=0}^{N-1} D_{m,n}W_{n,l} \\
&= \sum_{m,n=0}^{N-1} \overline{W_N(k,m)} W_N(n,l) \overline{W_N} D_{m,n} W_N \\
&= \frac{1}{N} \sum_{m,n=0}^{N-1} e^{2\pi j \frac{ln-km}{N}} \overline{W_N} D_{m,n} W_N, \\
\text{and thus } (W^{-1}DW)_{k,l}(q,r) &= \frac{1}{N} \sum_{m,n=0}^{N-1} e^{2\pi j \frac{ln-km}{N}} \sum_{s=0}^{N-1} \overline{W_N(q,s)} (D_{m,n}W_N)(s,r) \\
&= \frac{1}{N\sqrt{N}} \sum_{m,n,s=0}^{N-1} e^{2\pi j \frac{ln-km-qs}{N}} \sum_{t=0}^{N-1} D_{m,n}(s,t) W_N(t,r) \\
&= \frac{1}{N^2} \sum_{m,n,s,t=0}^{N-1} e^{2\pi j \frac{ln+rt-km-qs}{N}} D_{m-n,0}(s-t,0) \\
&= \frac{1}{N^2} \sum_{m,s=0}^{N-1} \sum_{n'=m-N+1}^{m} \sum_{t'=r-N+1}^{s} e^{2\pi j \frac{l(m-n')+r(s-t')-km-qs}{N}} D_{n',0}(t',0)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{N^2} \sum_{m,s=0}^{N-1} e^{2\pi j \frac{(l-k)m+(r-q)s}{N}} \sum_{n',t'=0}^{N-1} e^{2\pi j \frac{ln'+rt'}{N}} D(t'+n'N,0) \\
&= \frac{1}{N^2} \cdot N\delta(l-k) \cdot N\delta(r-q) \cdot \sum_{n',t'=0}^{N-1} e^{-2\pi j \frac{ln'+rt'}{N}} h(t',n') \\
&= N^2 DFT(h)(r,l)\delta(l-k)\delta(r-q).
\end{aligned}$$

Hence $W^{-1}DW$ is diagonal.
Thus the eigenvalues of D are the diagonal entries of $W^{-1}DW$, i.e. $\{N^2 DFT(h)(u,v) : 0 \le u,v \le N-1\}$.

(b) Since L is also block-circulant, it is also diagonalizable by W. Denote $W^{-1}DW$ by Λ_D and $W^{-1}LW$ by Λ_L. Then

$$\begin{aligned}
\lambda D^T D + L^T L &= \lambda D^* D + L^* L \\
&= \lambda (W\Lambda_D W^{-1})^* (W\Lambda_D W^{-1}) + (W\Lambda_L W^{-1})^* (W\Lambda_L W^{-1}) \\
&= \lambda W \Lambda_D^* \Lambda_D W^{-1} + W \Lambda_L^* \Lambda_L W^{-1}
\end{aligned}$$

and $\lambda D^T = \lambda D^* = \lambda(W\Lambda_D W^{-1})^* = \lambda W\Lambda_D^* W^{-1}$. Hence

$$
\begin{aligned}
(\lambda W\Lambda_D^*\Lambda_D W^{-1} + W\Lambda_L^*\Lambda_L W^{-1})\mathcal{S}(f) &= \lambda W\Lambda_D^* W^{-1}\mathcal{S}(g),\\
(\lambda\Lambda_D^*\Lambda_D + \Lambda_L^*\Lambda_L)\mathcal{S}(DFT(f)) &= \lambda\Lambda_D^*\mathcal{S}(DFT(g))
\end{aligned}
$$

and thus

$$
\begin{aligned}
DFT(f)(u,v) &= \frac{\lambda N^2\overline{DFT(h)(u,v)}}{N^4[\lambda|DFT(h)(u,v)|^2 + |DFT(p)(u,v)|^2]}DFT(g)(u,v)\\
&= \frac{1}{N^2}\frac{\lambda\overline{DFT(h)(u,v)}}{\lambda|DFT(h)(u,v)|^2 + |DFT(p)(u,v)|^2}DFT(g)(u,v).
\end{aligned}
$$

2. For any $i, k \in \mathbb{Z} \cap [0, N-1]$,

$$
\begin{aligned}
UC\bar{U}(i,k) &= \sum_{l=0}^{N-1} U(i,l)C\bar{U}(l,k)\\
&= \sum_{l,m=0}^{N-1} U(i,l)C(l,m)\overline{U(m,k)}\\
&= \frac{1}{N^2}\sum_{l,m=0}^{N-1} c_{l-m}e^{2\pi j\frac{km-il}{N}}\\
&= \frac{1}{N^2}\sum_{l=0}^{N-1}\sum_{m'=l+1-N}^{l} c_{m'}e^{2\pi j\frac{k(l-m')-il}{N}}\\
&= \frac{1}{N^2}\sum_{l,m'=0}^{N-1} c_{m'}e^{2\pi j\frac{l(k-i)-km'}{N}}\\
&= \frac{1}{N^2}\sum_{m'=0}^{N-1} c_{m'}e^{-km'}\cdot N\delta(k-i)\\
&= \frac{1}{N}\delta(k-i)\sum_{m'=0}^{N-1} c_{m'}e^{-km'}.
\end{aligned}
$$

Hence $UC\bar{U}$ is diagonal. Since $\sqrt{N}U$ is unitary, the eigenvalues of C are the diagonal entries of $(\sqrt{N}U)C(\sqrt{N}\bar{U})$, i.e. $\left\{\sum_{m=0}^{N-1} c_m e^{-km} : k = 0, 1, \cdots, N-1\right\}$.

3. $E(f) = \int_\Omega |f(x,y) - g(x,y)|^2 + \lambda\|\nabla f(x,y)\|^2\,dx\,dy$. Then for any $\varphi : \Omega \to \mathbb{R}$,

$$
\begin{aligned}
\frac{\partial E(f+t\varphi)}{\partial t}\Big|_{t=0} &= \frac{\partial}{\partial t}\Big|_{t=0}\int_\Omega \left\{|f(x,y) + t\varphi(x,y) - g(x,y)|^2 + \lambda\|\nabla(f+t\varphi)(x,y)\|^2\right\}\,dx\,dy\\
&= 2\int_\Omega \varphi(x,y)[f(x,y) - g(x,y)] + \lambda\frac{\partial}{\partial t}\Big|_{t=0}\int_\Omega [\|\nabla f(x,y)\|^2\\
&\quad + 2t\nabla f(x,y)\cdot\nabla\varphi(x,y) + t^2\|\nabla\varphi(x,y)\|^2]\,dx\,dy\\
&= 2\int_\Omega \left\{\varphi(x,y)[f(x,y) - g(x,y)] + \lambda\nabla f(x,y)\cdot\nabla\varphi(x,y)\right\}\,dx\,dy\\
&= 2\int_\Omega \{\varphi(x,y)[f(x,y) - g(x,y)]\}\,dx\,dy + 2\lambda\int_{\partial\Omega}\varphi(x,y)\nabla f(x,y)\cdot\vec{n}(x,y)\,ds\\
&\quad - 2\lambda\int_\Omega \left\{\varphi(x,y)[\nabla\cdot\nabla f(x,y)]\right\}\,dx\,dy.
\end{aligned}
$$

Hence a descent direction is:

$$
\varphi(x,y) = \begin{cases} -[f(x,y) - g(x,y)] + \lambda\nabla\cdot\nabla f(x,y) & \text{if } (x,y)\in\Omega\\ -\nabla f(x,y)\cdot\vec{n}(x,y) & \text{if } (x,y)\in\partial\Omega, \end{cases}
$$

and thus $E(f)$ can be iteratively minimized by updating f:

$$f^{n+1}(x,y) = \begin{cases} f^n(x,y) - \Delta t \left\{ [f^n(x,y) - g(x,y)] + \lambda \nabla \cdot \nabla f^n(x,y) \right\} & \text{if } (x,y) \in \Omega \\ f^n(x,y) - \Delta t \nabla f^n(x,y) \cdot \vec{n}(x,y) & \text{if } (x,y) \in \partial\Omega. \end{cases}$$

4.

$$E_{snake,2}(\gamma) = \int_0^{2\pi} \frac{1}{2}\|\gamma'(s)\|^2\, ds + \alpha \int_0^{2\pi} \frac{1}{2}\|\gamma''(s)\|^2\, ds + \beta \int_0^{2\pi} V(\gamma(s))\, ds,$$

Given a contour $\gamma^n(t)$ at the n-th iteration. We proceed to perturb $\gamma^{n+1}(t)$ by $\gamma^{n+1} := \gamma^n + \varepsilon\varphi$ to minimize $E_{snake,2}$. We need to find φ s.t. $\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} E_{snake,2}(\gamma^{n+1}) < 0$.

$$\begin{aligned}
&\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} E_{snake,2}(\gamma^{n+1}) \\
&= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \int_0^{2\pi} \frac{1}{2}\|(\gamma^n)'(s) + \varepsilon\varphi'(s)\|^2\ ds\ + \alpha \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \int_0^{2\pi} \frac{1}{2}\|(\gamma^n)''(s) + \varepsilon\varphi''(s)\|^2\ ds \\
&\quad + \beta \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \int_0^{2\pi} V(\gamma^n(s) + \varepsilon\varphi(s))\ ds \\
&= \int_0^{2\pi} (\gamma^n)'(s) \cdot \varphi'(s)\ ds\ + \alpha \int_0^{2\pi} (\gamma^n)''(s) \cdot \varphi''(s)\ ds\ + \beta \int_0^{2\pi} \nabla V(\gamma^n(s)) \cdot \varphi(s)\ ds \\
&= -\int_0^{2\pi} (\gamma^n)''(s) \cdot \varphi(s)\ ds\ + \alpha \int_0^{2\pi} (\gamma^n)^{(4)}(s) \cdot \varphi(s)\ ds\ + \beta \int_0^{2\pi} \nabla V(\gamma^n(s)) \cdot \varphi(s)\ ds \\
&= \int_0^{2\pi} [-(\gamma^n)''(s) + \alpha(\gamma^n)^{(4)}(s) + \beta \nabla V(\gamma^n(s))] \cdot \varphi(s)\ ds
\end{aligned}$$

In order that $\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} E_{snake,2}(\gamma^{n+1}) < 0$ (decreasing), we must have:

$$\varphi(s) = (\gamma^n)''(s) - \alpha(\gamma^n)^{(4)}(s) - \beta\nabla V(\gamma^n(s))$$

Thus, we must modify γ^n by:

$$\gamma^{n+1} = \gamma^n + \varepsilon((\gamma^n)''(s) - \alpha(\gamma^n)^{(4)}(s) - \beta\nabla V(\gamma^n(s))) \qquad \text{for small } \varepsilon > 0$$

or

$$\frac{\gamma^{n+1} - \gamma^n}{\varepsilon} = \underbrace{(\gamma^n)''(s) - \alpha(\gamma^n)^{(4)}(s) - \beta\nabla V(\gamma^n(s))}_{-\nabla E(\gamma^n(s))\ (definition)}$$

In the continuous setting, we aim to obtain a time-dependent contour: $\gamma_t(s) := \gamma(s;t)$ such that: $\frac{d}{dt}\gamma_t(s) = -\nabla E(\gamma_t(s))$.