Lecture 8: Eigenspace and diagonalization

Recall: (1) n distinct eigenvalues + $\dim(V) = n$ $\overset{\checkmark}{\Rightarrow}$ / $\not\Leftarrow$ Diagonalizable

(2) Diagonalizable $\overset{\checkmark}{\Rightarrow}$ / $\not\Leftarrow$ Char poly splits

(3) $\dim(\underline{E_\lambda}) = \dim(N(T - \lambda I))$ is important

Eigenspace (λ = eigenvalue)

$1 \le \dim(E_\lambda) \le m$ = multiplicity

Finding basis of eigenvectors

Theorem 1: Let $T: V \to V$ (finite dim). $\lambda_1, \lambda_2, \ldots, \lambda_k$ = distinct eigenvalues. Let S_i = finite lin. ind. subset of E_{λ_i}.

Then: $S_1 \cup S_2 \cup \ldots \cup S_k$ is lin. independent subset of V.

We need a lemma to prove it.

Lemma: Let $T: V \to V$, $\lambda_1, \lambda_2, \ldots, \lambda_k$ = distinct eigenvalues.

Let $\vec{v}_i \in E_{\lambda_i}$. If $\vec{v}_1 + \vec{v}_2 + \ldots + \vec{v}_k = \vec{0}$, then $\vec{v}_i = 0$ for all i.

Proof: Suppose not. By rearrangement, we can assume $\vec{v}_i \neq 0$ for $1 \le i \le m$ and $\vec{v}_i = 0$ for $i > m$.

Then: We have $\vec{v}_i$ = eigenvector of λ_i and $\vec{v}_1 + \ldots + \vec{v}_m = \vec{0}$.

But it is a contradiction since $\{\vec{v}_1, \ldots, \vec{v}_m\}$ must be lin. ind.

associated to distinct eigenvalues.

$\therefore \vec{v}_i = 0$ for all i.

Proof of Theorem 1:

Let $S_i = \{\vec{v}_1^{\,i}, \vec{v}_2^{\,i}, \ldots, \vec{v}_{n_i}^{\,i}\}$

Let $S = \bigcup_i S_i = \{v_j^{\,i} : 1 \le j \le n_i ;\ 1 \le i \le k\}$

Let $\sum_{i=1}^{k} \underbrace{\left(\sum_{j=1}^{n_i} a_{ij} \vec{v}_j^{\,i} \right)}_{\vec{w}_i} = \vec{0} \Rightarrow \vec{w}_1 + \vec{w}_2 + \ldots + \vec{w}_K = \vec{0}$.

Since $\vec{w}_i \in E_{\lambda_i}$, by lemma, $\vec{w}_i = 0$ for all i.

But S_i is lin independent and $\vec{w}_i = 0$

$\therefore \sum_{j=1}^{n_K} a_{ij} \vec{v}_j^{\,i} = 0 \Rightarrow a_{ij} = 0$ for all j.

We conclude S is lin. ind. subset of V.

Remark: • Theorem 2 $\Rightarrow$ if $\dim(E_{\lambda_1}) + \ldots + \dim(E_K) = \dim(V)$,

then the union of bases of E_{λ_i} = basis of V.

(In this case, T is diagonalizable.)

Condition for diagonalizable.

Theorem 2 (Important theorem):

Let $T: V \to V$ (finite-dim) such that char poly splits. Let $\lambda_1, \ldots, \lambda_K$ be distinct eigenvalues of T. Then:

(i) T diagonalizable $\Longleftrightarrow$ multiplicity of $\lambda_i = \dim(E_{\lambda_i})$ for all i.

(ii) T is diagonalizable, β_i = ordered basis for E_{λ_i} for each i,

then: $\beta = \beta_1 \cup \beta_2 \cup \ldots \cup \beta_n$ = ordered basis of V consisting of eigenvectors.

Proof: (i) Let m_i = multiplicity of λ_i; $d_i = \dim(E_{\lambda_i})$; $n = \dim(V)$

($\Rightarrow$) Suppose T is diagonalizable. Let β = basis of eigenvectors.

Let $\beta_i = E_{\lambda_i} \cap \beta$; Let n_i = # of elements in β_i.

Then: $n_i \leq d_i$ ($\because$ β_i = lin. independent subset of E_{λ_i})

Also, $\sum_{i=1}^{k} n_i = n$ (β has n elements; $\bigcup_i \beta_i = \beta$)

$\therefore n = \sum_{i=1}^{k} n_i \leq \sum_{i=1}^{k} d_i \leq \sum_{i=1}^{k} m_i = n$

(Sum of multiplicity $= n$)

($d_i \leq m_i$ by last time)

$$\Rightarrow \sum_{i=1}^{k} (m_i - d_i) \geq 0 \quad \text{and} \quad \underbrace{\sum_{i=1}^{k} (n_i - d_i)}_{= \sum_{i=1}^{k} (m_i - d_i)} \leq 0$$

$$\Rightarrow \sum_{i=1}^{k} (m_i - d_i) = 0$$

But $m_i - d_i \geq 0$ for all i, we have $m_i - d_i = 0$ for all i.

$\Rightarrow m_i = d_i$ for all i.

($\Leftarrow$) If $m_i = d_i$ for all i. Let β_i = ordered basis of E_{λ_i}.

Let $\beta = \beta_1 \cup \beta_2 \cup \dots \cup \beta_k$. By Theorem 1, β is lin. independent subset of V.

Now, $\left(\sum_{i=1}^{k} d_i\right)$ = # of elements in $\beta = \sum_{i=1}^{k} m_i = n$

$\therefore$ β = ordered basis of V of eigenvectors.

(ii) Simple consequence from the proof of (i).