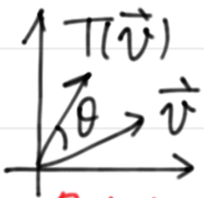
Lecture 6: Properties of eigenvectors

Last time:

- λ is an eigenvalue of T iff: $\det([T]_\beta - \lambda I) = 0$ for some ordered basis β
- $\vec{v}$ is an eigenvector of T associated to an eigenvalue λ iff: $\vec{v} \in N(T - \lambda I) := \{\vec{x} \in V : (T - \lambda I)(\vec{x}) = \vec{0}\}$

Example 1: Last time: $T : \mathbb{R}^2 \to \mathbb{R}^2$ where:

$$T(a_1, a_2) = (a_1 \cos\theta - a_2 \sin\theta,\ a_1 \sin\theta + a_2 \cos\theta)$$



$$[T]_\beta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Rotate counter-clockwisely by θ

standard ordered basis

To compute eigenvalues, consider:

$$f(t) = \det([T]_\beta - tI) = 0 \Leftrightarrow \det \begin{pmatrix} \cos\theta - t & -\sin\theta \\ \sin\theta & \cos\theta - t \end{pmatrix} = 0$$

$$\Leftrightarrow t^2 - 2t\cos\theta + 1 = 0$$

$$\Leftrightarrow t = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

But $4\cos^2\theta - 4 < 0$ if $0 < \theta < \pi$

Thus, T doesn't have eigenvalues/eigenvectors if $0 < \theta < \pi$.

Geometric interpretation of eigenvector/eigenvalues

Let $T: V \to V$, $(\vec{v}, \lambda)$ = eigenvector of T associated with eigenvalue λ.

Then: $T(\vec{v}) = \lambda \vec{v}$.

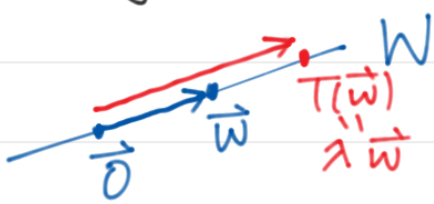
Consider the subspace: $W = \{c\vec{v} : c \in \mathbb{R}\} = \text{span}(\vec{v})$

For any $w \in W$, $\vec{w} = c\vec{v}$ for some $c \in \mathbb{R}$ and

$$T(\vec{w}) = T(c\vec{v}) = c\lambda\vec{v} = \lambda c\vec{v} = \lambda\vec{w}.$$

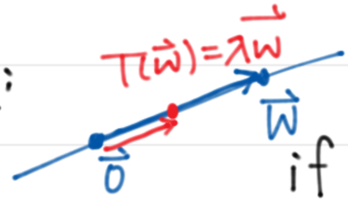
Geometrically,

Case 1:



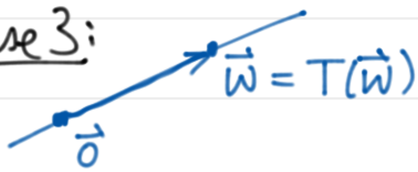
if $\lambda > 1$

Case 2:



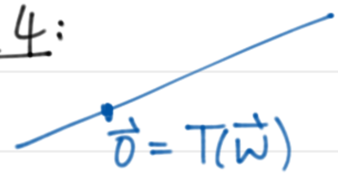
if $0 < \lambda < 1$

Case 3:



if $\lambda = 1$

Case 4:



if $\lambda = 0$

Case 5:

if $\lambda < 0$

Linear independency of eigenvectors

We observe that if $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ are eigenvectors of distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ are linearly independent.

We'll rigorously prove this observation.

Theorem 1: Let $T: V \to V$. Let $\lambda_1, \lambda_2, \dots, \lambda_k$ be distinct eigenvalues of T. If $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ are eigenvectors of $\lambda_1, \lambda_2, \dots, \lambda_k$, then $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ are linearly independent.

Proof: We will use mathematical induction on k.

For $k=1$, $\vec{v}_1 \neq \vec{0}$ by definition of eigenvector. Hence, $\{\vec{v}_1\}$ is linearly independent.

Now, assume that the thm is true for $k-1$ distinct eigenvalues.
$(k-1 \geq 1)$

Consider $\lambda_1, \lambda_2, \dots, \lambda_k = k$ distinct eigenvalues. We need to show $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ are linearly independent.

Consider: $a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_k\vec{v}_k = \vec{0}$ —— $(*)$

$$\Rightarrow (T - \lambda_k I)(a_1\vec{v}_1 + \dots + a_k\vec{v}_k) = \vec{0}$$

$$\Rightarrow a_1(\underbrace{T(\vec{v}_1)}_{\lambda_1\vec{v}_1} - \lambda_k\vec{v}_1) + \dots + a_k(\underbrace{T(\vec{v}_k)}_{\lambda_k\vec{v}_k} - \lambda_k\vec{v}_k) = \vec{0}$$

$$\Rightarrow a_1(\lambda_1 - \lambda_k)\vec{v}_1 + \dots + a_{k-1}(\lambda_{k-1} - \lambda_k)\vec{v}_{k-1} = \vec{0}$$

By induction hypothesis, $\{\vec{v}_1, \dots, \vec{v}_{k-1}\}$ are linearly independent.

Thus, $a_1(\lambda_1 - \lambda_k) = \dots = a_{k-1}(\lambda_{k-1} - \lambda_k) = 0$

Since $(\lambda_i - \lambda_k) \neq 0$ for $1 \leq i \leq k-1$, we have: $a_1 = \dots = a_{k-1} = 0$.

Thus, $(*)$ is reduced to $a_k\vec{v}_k = \vec{0}$. $\therefore a_k = 0$.

Thus, $\{\vec{v}_1, \dots, \vec{v}_k\}$ are linearly independent.