Lecture 22: Orthogonal projection

Observation: Let V be a finite-dimensional inner product space. Let W be a subspace of V.

Let T be the orthogonal projection of V onto W.

[Recall that: $V = W \oplus W^\perp$. An orthogonal projection T does the following: $T(\underset{\underset{W}{\wedge}}{w_1} + \underset{\underset{W^\perp}{\wedge}}{w_2}) = w_1$]

Let β be an orthonormal basis $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ of V such that $W = \text{Span}(\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\})$.

Then:
$$[T]_\beta = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$$

Remark: $V = W \oplus W^\perp = R(T) \oplus R(T)^\perp = R(T) \oplus N(T)$

Theorem 1: (The Spectral Theorem) Suppose T is a linear operator on a finite dimensional inner product space V over $\mathbb{F}$, with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. Assume that T is normal if $\mathbb{F} = \mathbb{C}$ and T is self-adjoint if $\mathbb{F} = \mathbb{R}$.

Let W_i = eigenspace of T corresponding to λ_i.

T_i = orthogonal projection of V onto W_i.

Then: (1) $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$

(2) If $W_i' = \bigoplus_{j \neq i} W_j = W_1 \oplus \ldots \oplus W_{i-1} \oplus W_{i+1} \oplus \ldots \oplus W_k$

then: $W_i^\perp = W_i'$

③ $T_i T_j = \delta_{ij} T_i$ for $1 \le i, j \le k$

$[\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \ne j. \end{cases}$

④ $I = T_1 + T_2 + \dots + T_k$

⑤ $T = \lambda_1 T_1 + \lambda_2 T_2 + \dots + \lambda_k T_k$

Proof: ① V has an orthonormal basis of eigenvectors of T.

Hence, $V = W_1 \oplus W_2 \oplus \dots \oplus W_k$

② Let $x \in W_i$, $y \in W_j$ $(j \ne i)$. Then:

$\langle x, y \rangle = 0$ (since distinct eigenvectors are orthogonal for normal T)

Thus, $W_i' \subseteq W_i^\perp$

Now, $\dim(W_i') = \sum_{j \ne i} \dim(W_j) = \dim(V) - \dim(W_i)$

Also, $\dim(W_i^\perp) = \dim(V) - \dim(W_i)$

$[$as $\dim(W) + \dim(W_i) = \dim(V)]$

Hence, $W_i' = W_i^\perp$.

③ Follows from the definition of orthogonal projection.

④ For any $x \in V$, $x = x_1 + x_2 + \dots + x_k$

where $x_i \in W_i$

Need to prove: $x_i \in T_i(x)$

Now, $N(T_i) = R(T_i)^\perp = W_i^\perp = W_i'$.

$\therefore T_i\left(\sum_{j \ne i} x_j\right) = 0$

Hence, $T_i(x) = T_i(x_i + \sum_{j \neq i} x_j) = T_i(x_i) = x_i$

⑤ For any $x \in V$, write:

$$x = x_1 + x_2 + \dots + x_k \quad \text{where} \quad x_i \in W_i.$$

Then:
$$\begin{aligned} T(x) &= T(x_1) + T(x_2) + \dots + T(x_k) \\ &= \lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_k x_k \\ &= \lambda_1 T_1(x) + \lambda_2 T_2(x) + \dots + \lambda_k T_k(x) \\ &= (\lambda_1 T_1 + \dots + \lambda_k T_k)(x) \end{aligned}$$

<u>Remark:</u>

- $\{\lambda_1, \lambda_2, \dots, \lambda_k\}$ is called the <u>spectrum</u> of T.
- $I = T_1 + T_2 + \dots + T_k$ is called the <u>resolution of the identity operator induced by T</u>.
- $T = \lambda_1 T_1 + \lambda_2 T_2 + \dots + \lambda_k T_k$ is called the <u>spectral decomposition of T</u>.
- The spectral decomposition of T is unique up to the ordering of eigenvalues.
- Let β = union of orthonormal bases of W_i's.

  $m_i = \dim(W_i)$.

  Then: $$[T]_\beta = \begin{pmatrix} \lambda_1 I_{m_1} & & & \\ & \lambda_2 I_{m_2} & & \\ & & \ddots & \\ & & & \lambda_k I_{m_k} \end{pmatrix}$$
- Also, if $T = \lambda_1 T_1 + \dots + \lambda_k T_k$

  then $g(T) = g(\lambda_1) T_1 + \dots + g(\lambda_k) T_k$

  for any polynomial g. (Check)

Interesting consequence of the spectral theorem

Corollary 1: If $\mathbb{F} = \mathbb{C}$, then T is normal if and only if $T^* = g(T)$ for some polynomial g.

Proof: ($\Leftarrow$). Obvious since polynomial in T commutes with T.

($\Rightarrow$) Suppose T is normal. Let $T = \lambda_1 T_1 + \dots \lambda_k T_k$ be the spectral decomposition of T. Then:

$$T^* = \overline{\lambda_1}\, T_1^* + \dots + \overline{\lambda_k}\, T_k^* = \overline{\lambda_1}\, T_1 + \dots + \overline{\lambda_k}\, T_k$$

($\because$ T_i is self-adjoint)

Using the Lagrange interpolation formula, we can find g such that $g(\lambda_i) = \overline{\lambda_i}$ for $1 \leq i \leq k$.

Then:
$$g(T) = g(\lambda_1) T_1 + \dots + g(\lambda_k) T_k$$
$$= \overline{\lambda_1}\, T_1 + \dots + \overline{\lambda_k}\, T_k = T^*.$$

Corollary 2: If $\mathbb{F} = \mathbb{C}$, then T is unitary if and only if T is normal and $|\lambda| = 1$ for every eigenvalues λ of T.

Proof: ($\Rightarrow$) If T is unitary, then T is normal and every eigenvalues of T has absolute value $= 1$.

($\Leftarrow$) Let $T = \lambda_1 T_1 + \dots + \lambda_k T_k$ (since T is normal)

If $|\lambda| = 1$ for all eigenvalues of T, then

$$TT^* = (\lambda_1 T_1 + \dots + \lambda_k T_k)(\overline{\lambda_1}\, T_1 + \dots + \overline{\lambda_k}\, T_k)$$
$$= |\lambda_1|^2 \underbrace{T_1^2}_{=T_1} + \dots + |\lambda_k|^2 \underbrace{T_k^2}_{=T_k} = T_1 + \dots + T_k = I.$$

$\therefore$ $TT^* = T^*T = I.$

Lagrange Interpolation formula

Goal: Let $c_0, c_1, \dots, c_n \in \mathbb{F}$ be distinct scalars.
Find polynomial g such that $g(c_i) = b_i$ for given $b_0, b_1, \dots, b_n \in \mathbb{F}$.

Method: Let $f_i = \dfrac{(x-c_0)\cdots(x-c_{i-1})(x-c_{i+1})\cdots(x-c_n)}{(c_i-c_0)\cdots(c_i-c_{i-1})(c_i-c_{i+1})\cdots(c_i-c_n)}$

Then: $g = \sum_{i=0}^{n} b_i f_i$

Example: Let $c_0 = 1$, $c_1 = 2$, $c_2 = 3$
$b_0 = 8$, $b_1 = 5$, $b_2 = -4$

Then: $f_0 = \dfrac{(x-2)(x-3)}{(1-2)(1-3)} = \frac{1}{2}(x^2 - 5x + 6)$

$f_1 = \dfrac{(x-1)(x-3)}{(2-1)(2-3)} = (-1)(x^2 - 4x + 3)$

$f_2 = \dfrac{(x-1)(x-2)}{(3-1)(3-2)} = \frac{1}{2}(x^2 - 3x + 2)$

$\therefore\ g(x) = 8f_0 + 5f_1 - 4f_2$
$= -3x^2 + 6x + 5$

Corollary 3: If $\mathbb{F} = \mathbb{C}$ and T is normal, then T is self-adjoint if and only if every eigenvalues of T are real

Proof: Let $T = \lambda_1 T_1 + \cdots + \lambda_k T_k$ (Spectral decomposition of T since T is normal)

($\Leftarrow$) Assume all eigenvalues are real.

Then $T^* = \overline{\lambda_1} T_1 + \dots + \overline{\lambda_k} T_k$

$= \lambda_1 T_1 + \dots + \lambda_k T_k = T$

($\Rightarrow$) T is self-adjoint.

$\therefore$ all eigenvalues are real has been proven before.

<u>Corollary 4:</u> Let $T = \lambda_1 T_1 + \dots + \lambda_k T_k$

(assume T is normal for $\mathbb{F} = \mathbb{C}$ and

T is self-adjoint for $\mathbb{F} = \mathbb{R}$)

Then: each T_j is a polynomial in T.

<u>Proof:</u> Let g_j be a polynomial $(1 \leq j \leq k)$ such that

$$g_j(\lambda_i) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

Then: $g_j(T) = g_j(\lambda_1) T_1 + \dots + g_j(\lambda_k) T_k$

$= \delta_{1j} T_1 + \dots + \delta_{kj} T_k = T_j$.

<u>Formal definition of projection</u>

<u>Definition:</u> (Projection) Let $V = W_1 \oplus W_2$ where W_1 and W_2 are subspaces. $T: V \to V$ is called the <u>projection on W_1 along W_2</u> if for $x = x_1 + x_2$ with $x_1 \in W_1$ and $x_2 \in W_2$, we have $T(x) = x_1$.

Note: $T^2 = T$

<u>Definition:</u> (Orthogonal projection) Let V be an inner product space and let $T: V \to V$ be a projection. We say that T is an <u>orthogonal projection</u> if $R(T)^{\perp} = N(T)$ and $N(T)^{\perp} = R(T)$.