Lecture 18: More about normal operator

Example: Symmetric real matrix is normal ($A^T = A$)

because: $\underset{\underset{A^*A}{\parallel}}{A^TA} = A^2 = \underset{\underset{AA^*}{\parallel}}{AA^T}$

Skew-symmetric real matrix is normal ($A^T = -A$)

because $A^*A = A^TA = -A^2 = AA^T = AA^*$.

Observation: Let $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ (Rotation on $\mathbb{R}^2$ over $\mathbb{F} = \mathbb{R}$)

Then A does not have eigenvector if $0 < \theta < \pi$

A is normal. $\therefore$ Normal $\not\Rightarrow$ existence of orthonormal basis over $\mathbb{F} = \mathbb{R}$

But we will prove Normal $\Rightarrow$ orthonormal basis over $\mathbb{F} = \mathbb{C}$.

Theorem: Let V = inner product space. Let T be a normal operator on V. Then:

(a) $\|T(\vec{x})\| = \|T^*(\vec{x})\|$ for all $\vec{x} \in V$.

(b) $(T - cI)$ is normal for $\forall c \in \mathbb{F}$.

(c) If $\vec{x}$ = eigenvector of T, then $\vec{x}$ is an eigenvector of T^*.
$\left(T(\vec{x}) = \lambda \vec{x} \Rightarrow T^*(\vec{x}) = \bar{\lambda} \vec{x} \right)$

(d) If λ_1 and λ_2 are distinct eigenvalues of T with eigenvectors $\vec{x}_1$ and $\vec{x}_2$, then $\vec{x}_1$ and $\vec{x}_2$ are orthogonal.

<u>Proof:</u> (a) For all $\vec{x} \in V$,

$$\|T(\vec{x})\|^2 = \langle T(\vec{x}), T(\vec{x}) \rangle = \langle \vec{x}, T^*T(\vec{x}) \rangle$$
$$= \langle \vec{x}, TT^*(\vec{x}) \rangle$$
$$= \langle T^*(\vec{x}), T^*(\vec{x}) \rangle = \|T^*(\vec{x})\|^2$$

(b) $$(T - cI)^*(T - cI) = (T^* - \bar{c}I)(T - cI)$$
$$= \underset{\overset{\shortparallel}{TT^*}}{T^*T} - cT^* - \bar{c}T - \bar{c}cI$$
$$= (T - cI)(T - cI)^*$$

(c) Let $T(\vec{x}) = \lambda \vec{x}$ for some $\vec{x} \in V$.

Then: $$0 = \|(T - \lambda I)(\vec{x})\| = \|(T - \lambda I)^*(\vec{x})\|$$
$$= \|(T^* - \bar{\lambda} I)(\vec{x})\|$$

$\therefore \quad T^*(\vec{x}) = \bar{\lambda}\vec{x}$.

(d) Let λ_1 and λ_2 be two distinct eigenvalues with eigenvectors $\vec{x}_1$ and $\vec{x}_2$ respectively. Then:

$$\lambda_1 \langle \vec{x}_1, \vec{x}_2 \rangle = \langle \lambda_1 \vec{x}_1, \vec{x}_2 \rangle = \langle T(\vec{x}_1), \vec{x}_2 \rangle$$
$$= \langle \vec{x}_1, T^*(\vec{x}_2) \rangle$$
$$= \langle \vec{x}_1, \bar{\lambda}_2 \vec{x}_2 \rangle = \lambda_2 \langle \vec{x}_1, \vec{x}_2 \rangle$$

$\therefore \quad \underbrace{(\lambda_1 - \lambda_2)}_{\neq 0} \langle \vec{x}_1, \vec{x}_2 \rangle = 0 \Rightarrow \langle \vec{x}_1, \vec{x}_2 \rangle = 0$.

<u>Theorem:</u> Let $T: V \to V$ be linear operator on an inner product space V (finite-dimensional over $\mathbb{F} = \mathbb{C}$). Then: T is normal iff $\exists$ orthonormal basis for V consisting of eigenvectors of T.

Proof: Suppose T is normal. The char poly splits over $\mathbb{F} = \mathbb{C}$ (fundamental thm of Algebra). By Schur's theorem, $\exists$ orthonormal basis $\beta = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ such that $[T]_\beta$ is upper-triangular. Then: $\vec{v}_1$ is an eigenvector of T.

$\left([T]_\beta = \begin{pmatrix} * & \\ 0 & \\ \vdots & * \\ 0 & \end{pmatrix} . \quad \text{Then}: \quad T(\vec{v}_1) = * \, \vec{v}_1 \right)$

We'll prove that all $\vec{v}_1, \dots, \vec{v}_n$ are actually eigenvectors by M.I.

For $n=1$, true.

Assume $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{k-1}$ are eigenvectors of T, we'll prove that $\vec{v}_k$ is also eigenvector of T.

For $j < k$, let λ_j be the eigenvalue associated to $\vec{v}_j$.

So, $T^*(\vec{v}_j) = \overline{\lambda_j} \vec{v}_j$.

Since A is upper-triangular,

$$T(\vec{v}_k) = A_{1k}\vec{v}_1 + A_{2k}\vec{v}_2 + \dots + A_{kk}\vec{v}_k.$$

Recall that $A_{jk} = \langle T(\vec{v}_k), \vec{v}_j \rangle$ since β is orthonormal.

$$\therefore A_{jk} = \langle T(\vec{v}_k), \vec{v}_j \rangle = \langle \vec{v}_k, T^*(\vec{v}_j) \rangle = \langle \vec{v}_k, \overline{\lambda_j} \vec{v}_j \rangle$$

$$= \lambda_j \underbrace{\langle \vec{v}_k, \vec{v}_j \rangle}_{0} = 0$$

for $j < k$.

$\therefore T(\vec{v}_k) = A_{kk}\vec{v}_k \implies \vec{v}_k$ is an eigenvector.

By M.I., all vectors in β are eigenvectors.

The converse is trivial.

(Shown before)

<u>Remark:</u> • The above thm is only true for <u>finite dimensional</u> over <u>$\mathbb{F} = \mathbb{C}$</u>.

<u>Theorem is invalid for infinite dim V</u>

<u>Example:</u> Consider $H = C([0, 2\pi])$.

Recall the inner product on H is defined as:

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(t)\, \overline{g(t)}\, dt.$$

Let $S = \{ f_n(t) = e^{int} = \cos nt + i \sin nt : n \in \mathbb{Z} \}$.

Recall that S is orthonormal.

Let $V = \text{Span}(S)$, which is an infinite dimensional inner product space.

Define $T : V \to V$ and $U : V \to V$ as follows:

$$T(f) = f_1 f \quad \text{and} \quad U(f) = f_{-1} f$$

Then: $T(f_n) = f_1 f_n = e^{i(n+1)t} = f_{n+1}$ for $\forall n$.

$U(f_n) = f_{-1} f_n = e^{i(n-1)t} = f_{n-1}$

So, $\langle T(f_m), f_n \rangle = \langle f_{m+1}, f_n \rangle = \delta_{m+1, n}$

$= \delta_{m, n-1}$

$= \langle f_m, f_{n-1} \rangle$

$= \langle f_m, U(f_n) \rangle$

$\left(\delta_{m,n} = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{otherwise} \end{cases} \right)$

$\therefore T^* = U$.

Also, $TT^* = TU = I$

$(\because TT^*(f) = TU(f) = T(f_{-1} f) = f_1 f_{-1} f = f)$

Similarly, $T^*T = UT = I$

$\therefore \quad T^*T = TT^* \Rightarrow T$ is normal.

But T has no eigenvectors. Suppose f is an eigenvector of T. Let $T(f) = \lambda f$ and let

$$f = \sum_{i=n}^{m} a_i f_i \quad \text{(where } a_m \neq 0\text{)}$$

Then: $\lambda f = \sum_{i=n}^{m} \lambda a_i f_i = T(f) = \sum_{i=n}^{m} a_i T(f_i)$

$$= \sum_{i=n}^{m} a_i f_{i+1}$$

Since $a_m \neq 0$, f_{m+1} can be written as a linear combination of $f_n, f_{n+1}, \dots, f_m$.

Contradicting that S is linearly independent.

<u>Definition:</u> Let T be linear operator on an inner product space V. Then: T is called <u>self-adjoint</u> (hermitian) iff $T = T^*$.

A matrix $A \in M_{n\times n}(\mathbb{F})$ is called <u>self-adjoint</u> (hermitian) iff $A = A^*$.

<u>Goal:</u> Self-adjoint $\rightarrow$ Real inner product space has orthonormal basis of eigenvectors.