Lecture 15: G-S process and Orthogonal complement

G-S process: produce orthonormal basis

Goal: Let $V = P_n(\mathbb{R})$. Can we get orthonormal basis for $P_n(\mathbb{R})$ with respect to some (weighted) inner product?

Important applications in physics!

e.g. Legendre polynomial, Chebyshev polynomial, Laguerre ...

Definition 1: Let $V = P_n(\mathbb{R})$. Let $w: \mathbb{R} \to \mathbb{R}$ such that $w(x) > 0$.

Then the <u>weighted inner product</u> is defined as:

$$\langle f, g \rangle = \int_{-1}^{1} w(x) f(x) g(x)\, dx.$$

Using G-S process, we can obtain orthonormal basis with respect to different weight. e.g. $w(x) = 1 \leftrightarrow$ Legendre; $w(x) = \frac{1}{\sqrt{1-x^2}} \leftrightarrow$ Chebyshev

Example 1: (Legendre polynomials) Let $V = P_n(\mathbb{R})$. Define:

$$\langle f(x), g(x) \rangle = \int_{-1}^{1} f(t) g(t)\, dt \quad (w(x) = 1)$$

Clearly, $\beta = \{1, x, x^2, \ldots, x^n\}$ = basis for V. We can obtain orthogonal polynomials from β using G-S process.

G-S process: $p_0(x) = 1$,

$$\left.\begin{aligned} p_1(x) &= x - \frac{\langle x, p_0 \rangle}{\langle p_0, p_0 \rangle} p_0(x) \\ p_2(x) &= x^2 - \frac{\langle x^2, p_0 \rangle}{\langle p_0, p_0 \rangle} p_0(x) - \frac{\langle x^2, p_1 \rangle}{\langle p_1, p_1 \rangle} p_1(x) \\ &\vdots \end{aligned}\right\} \text{Orthogonal polynomials}$$

$$p_n(x) = x^n - \frac{\langle x^n, p_0 \rangle}{\langle p_0, p_0 \rangle} p_0(x) - \cdots - \frac{\langle x^n, p_{n-1} \rangle}{\langle p_{n-1}, p_{n-1} \rangle} p_{n-1}(x)$$

Note that : $\langle x^n, x^m \rangle = 0$ if $n+m$ is odd

$\therefore\ p_0(x) = 1,\ p_1(x) = x - 0 = x\ ;\ p_2(x) = x^2 - \frac{\langle x^2, 1\rangle}{\langle 1, 1\rangle} 1 - 0 = x^2 - \frac{1}{3}.$

$$p_3(x) = x^3 - 0 - \frac{\langle x^3, x\rangle}{\langle x, x\rangle} x - 0 = x^3 - \frac{3}{5}x$$

$$p_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$\{p_0, p_1, \ldots, p_n\}$ are called <u>Legendre's polynomials</u>.

It becomes orthonormal polynomial by normalization:

$\left\{ \frac{p_0}{\|p_0\|}, \ldots, \frac{p_n}{\|p_n\|} \right\}$ (exercise)

<u>Example 2</u>: (Chebyshev polynomial) Let $V = P_n(\mathbb{R})$. Define:

$\langle f, g\rangle = \int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} f(t) g(t)\, dt.$ $\left(w(x) = \frac{1}{\sqrt{1-x^2}} \right)$

Again, we G-S process to obtain orthogonal basis from $\{1, x, \ldots, x^n\}$.

$$p_0(x) = 1,\ p_1(x) = x - \frac{\langle x, p_0\rangle}{\langle p_0, p_0\rangle} p_0(x) = x - \frac{\overset{0}{\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} t\, dt}}{\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}}\, dt} 1 = x$$

$$p_2(x) = x^2 - \frac{\langle x^2, p_0\rangle}{\langle p_0, p_0\rangle} p_0(x) - \frac{\langle x^2, p_1\rangle}{\langle p_1, p_1\rangle} p_1(x) = x^2 - \frac{1}{2}$$

$\vdots$

$p_n(x) = \cdots$

In fact, we can prove that : $p_{n+1}(x) = 2x\, p_n(x) - p_{n-1}(x)$.

<u>Remark</u>: Same procedure can be applied to different weight function to get different orthogonal polynomials.

e.g. $w(x) = 1$ (Legendre) ; $w(x) = \frac{1}{\sqrt{1-x^2}}$ (Chebyshev, 1st kind)

$w(x) = \sqrt{1-x^2}$ (Chebyshev, 2nd kind) ; $w(x) = e^{-x}$ (Laguerre)

$w(x) = (1+x)^{\alpha}(1-x)^{\beta}$ (Jacobi) etc

Definition 2: Let β = orthonormal subset (can be infinite) of V = inner product space. Let $x \in V$. Define the Fourier coefficients of x relative to β to be scalars $\langle x, y\rangle$ where $y \in \beta$.

Remark: More specifically, Fourier studied:

$$a_n = \int_0^{2\pi} f(t)\sin nt\, dt; \quad b_n = \int_0^{2\pi} f(t)\cos nt\, dt$$ or more generally,

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(t) e^{-int} dt$$

Recall: $S = \{e^{int}\}_{n\in\mathbb{Z}}$ is an orthonormal basis w.r.t.

$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f\bar{g}\, dt$. $\therefore$ c_n is the n-th Fourier coefficient of f relative to S,

since $\langle f, e^{int}\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(t) \underbrace{e^{-int}}_{\overline{e^{int}}} dt$

Example 3: Let $S = \{e^{int} : n \in \mathbb{Z}\}$ = orthonormal set in $V = C([0,2\pi])$

Compute the n-th Fourier coefficient of t relative to S:

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} t e^{-int} dt = -\frac{1}{in} \text{ (by integration by part) } (n \neq 0)$$

For $n=0$, $c_0 = \frac{1}{2\pi}\int_0^{2\pi} t\, dt = \pi$

We can prove that: $\|x\|^2 \geq \sum_{i=1}^{n} |\langle x, v_i\rangle|^2$ where: $\{v_1, v_2, \ldots, v_n\}$ = orthonormal subset.

$$\therefore \|f\|^2 \geq \sum_{n=-k}^{-1} |\langle f, f_n\rangle|^2 + |\langle f, 1\rangle|^2 + \sum_{n=1}^{k} |\langle f, f_n\rangle|^2$$

$$= \sum_{n=-k}^{-1} \frac{1}{n^2} + \pi^2 + \sum_{n=1}^{k} \frac{1}{n^2} = 2\sum_{n=1}^{k}\frac{1}{n^2} + \pi^2 \text{ for } \forall k.$$

Now, $\|f\|^2 = \frac{1}{2\pi}\int_0^{2\pi} t^2 dt = \frac{4}{3}\pi^2$, we get:

$$\frac{4}{3}\pi^2 \geq 2\sum_{n=1}^{k}\frac{1}{n^2} + \pi^2 \Rightarrow \sum_{n=1}^{k}\frac{1}{n^2} \leq \frac{\pi^2}{6} \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} \leq \frac{\pi^2}{6}.$$

(for $\forall k$)

Orthogonal complement

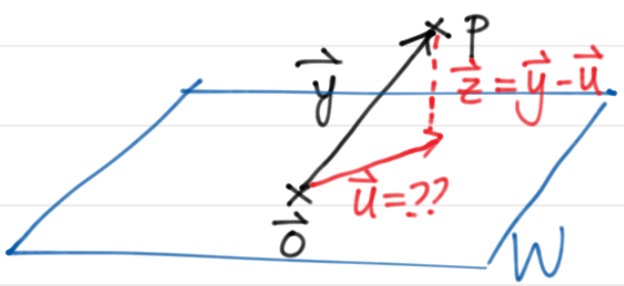
Definition 3: Let S = non-empty subset in V (inner product space). Define $S^{\perp}$ (S "perp") = set of all $\vec{v} \in V$ that are orthogonal to $\forall \vec{w} \in S$. (i.e. $S^{\perp} = \{\vec{x} \in V : \langle \vec{x}, \vec{y} \rangle = 0 \text{ for } \forall \vec{y} \in S\}$)

$S^{\perp}$ is called orthogonal complement of S.

Remark: (Trivial examples)

- $\{\vec{0}\}^{\perp} = V$ • $V^{\perp} = \{\vec{0}\}$ • $V = \mathbb{R}^3$, $\{e_1\}^{\perp} = y$-z plane $= \{(0, y, z) : y, z \in \mathbb{R}\}$

Application: Shortest distant from P to a plane



Find $\vec{u} \in W$ such that $\|\vec{y} - \vec{u}\|$ is smallest.

Theorem 1: Let W = finite-dim subspace of inner product space V. Let $\vec{y} \in V$. Then there exists an unique $\vec{u} \in W$ and an unique $\vec{z} \in W^{\perp}$ such that: $\vec{y} = \vec{z} + \vec{u}$.

Also, if $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ is an orthonormal basis for W, then

$$\vec{u} = \sum_{i=1}^{k} \langle \vec{y}, \vec{v}_i \rangle \vec{v}_i$$ (Orthogonal projection of $\vec{y}$ on W)

Proof: Let $\{\vec{v}_1, \dots, \vec{v}_k\}$ = orthonormal basis for W.

Let $\vec{u} = \sum_{i=1}^{k} \langle \vec{y}, \vec{v}_i \rangle \vec{v}_i$.

We need to show $\vec{z} = \vec{y} - \vec{u} \in W^\perp$. Only need to show $\langle \vec{z}, \vec{v}_j \rangle = 0$ for $\forall j$.

Now, $\langle \vec{z}, \vec{v}_j \rangle = \langle (\vec{y} - \sum_{i=1}^{k} \langle \vec{y}, \vec{v}_i \rangle \vec{v}_i), \vec{v}_j \rangle = \langle \vec{y}, \vec{v}_j \rangle - \sum_{i=1}^{k} \langle \vec{y}, \vec{v}_i \rangle \langle \vec{v}_i, \vec{v}_j \rangle$

$= \langle \vec{y}, \vec{v}_j \rangle - \langle \vec{y}, \vec{v}_j \rangle \underbrace{\langle \vec{v}_j, \vec{v}_j \rangle}_{1}$

For uniqueness of $\vec{u}$ and $\vec{z}$, suppose $\vec{y} = \vec{u} + \vec{z} = \underbrace{\vec{u}'}_{\in W} + \underbrace{\vec{z}'}_{\in W^\perp}$.

Then: $\vec{u} - \vec{u}' = \vec{z}' - \vec{z} \in W \cap W^\perp = \{\vec{0}\}$ (because if $\vec{w} \in W$ and $\vec{w} \in W^\perp$, then $\langle \vec{w}, \vec{w} \rangle = 0 \Rightarrow \vec{w} = \vec{0}$)

$\therefore$ $\vec{u} = \vec{u}'$ and $\vec{z} = \vec{z}'$.

Corollary 1: "$\vec{u}$ is the unique vector in W closest to $\vec{y} \in V$"

That is, for $\forall \vec{x} \in W$, $\|\vec{y} - \vec{x}\| \geq \|\vec{y} - \vec{u}\|$.

Also, equality holds $\Leftrightarrow$ $x = u$.

Proof: Let $\vec{x} \in W$. Then $\underset{\in W}{\vec{u}} - \underset{\in W}{\vec{x}} \in W$ is orthogonal to $z \in W^\perp$.

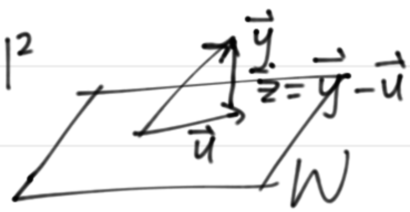
Note that $\|\vec{x} + \vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2$ if $\vec{x}, \vec{y}$ are orthogonal (exercise)

So, $\|\vec{y} - \vec{x}\|^2 = \|\vec{u} + \vec{z} - \vec{x}\|^2 = \|(\vec{u} - \vec{x}) + \vec{z}\|^2 = \|\underbrace{(\vec{u} - \vec{x})}_{\in W} + \underbrace{\vec{z}}_{\in W^\perp}\|^2$

$= \|\vec{u} - \vec{x}\|^2 + \|\vec{z}\|^2 \geq \|\vec{z}\|^2 = \|\vec{y} - \vec{u}\|^2$

If $\|\vec{y} - \vec{x}\| = \|\vec{y} - \vec{u}\|$; then: $\|\vec{u} - \vec{x}\|^2 + \|\vec{z}\|^2 = \|\vec{z}\|^2$

$\Rightarrow \|\vec{u} - \vec{x}\|^2 = 0 \Rightarrow \vec{x} = \vec{u}$.



Conversely, if $\vec{x} = \vec{u}$, then equality holds obviously.

Example 4: Let $V=\mathbb{R}^4$. Let $\beta = \{\underbrace{\frac{1}{\sqrt{2}}(1,1,0,0)}_{=\vec{v}_1}, \underbrace{\frac{1}{\sqrt{3}}(1,-1,1,0)}_{=\vec{v}_2}, \underbrace{\frac{1}{\sqrt{6}}(-1,1,2,0)}_{=\vec{v}_3}\}$

Let $W = \text{span}(\beta)$. β = orthonormal basis of W.

The orthogonal projection of $(2,3,4,5)$ to W is:

$$\vec{w}_0 = \langle \vec{w}, \vec{v}_1 \rangle \vec{v}_1 + \langle \vec{w}, \vec{v}_2 \rangle \vec{v}_2 + \langle \vec{w}, \vec{v}_3 \rangle \vec{v}_3$$

$$= \frac{5}{\sqrt{2}} \vec{v}_1 + \frac{3}{\sqrt{3}} \vec{v}_2 + \frac{9}{\sqrt{6}} \vec{v}_3.$$