Eigenvalue Problem

Recall: Convergence of iterative scheme:

$N\vec{x}^{k+1} = P\vec{x}^{k} + \vec{f}$ depends on the spectral radius $\rho(N^{-1}P)$.

$\therefore$ Need: numerical method to compute eigenvalues.

Computation of Spectral radius

Goal: Find eigenvalues with largest magnitude $\leftarrow$ Spectral radius

Two methods:

1. Power method
2. QR method

1. Power method

Let $A \in M_{n\times n}(\mathbb{C})$ with n eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ with eigenvectors $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$.

Let $X = \begin{pmatrix} | & | & & | \\ \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \\ | & | & & | \end{pmatrix} \in M_{n\times n}(\mathbb{C})$. Then, we know:

$$AX = X \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}.$$

Assuming: $|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|$

We'll use Power method to compute $|\lambda_1|$.

Observation: Start with an initial vector $\vec{x}^{(0)}$.

Consider the iterative scheme: $\vec{x}^{(k+1)} = \dfrac{A\vec{x}^{(k)}}{\|A\vec{x}^{(k)}\|_\infty}$ for $k=0,1,\ldots$

Suppose A is diagonalizable. That's, we can assume $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ form a basis for $\mathbb{C}^n$.

Take $\vec{x}^{(0)} = a_1\vec{x}_1 + a_2\vec{x}_2 + \cdots + a_n\vec{x}_n$ (assuming $a_1 \neq 0$)

Note that $A^k \vec{x}^{(0)} = a_1\lambda_1^k \left[\vec{x}_1 + \sum_{j=2}^{n} \frac{a_j}{a_1}\left(\frac{\lambda_j}{\lambda_1}\right)^k \vec{x}_j \right]$.

Hence, $\vec{x}^{(k)} = \dfrac{A\vec{x}^{(k-1)}}{\|A\vec{x}^{(k-1)}\|_\infty} = \dfrac{A\left(A\vec{x}^{(k-2)}/\|A\vec{x}^{(k-2)}\|_\infty\right)}{\|A\left(A\vec{x}^{(k-2)}/\|A\vec{x}^{(k-2)}\|_\infty\right)\|_\infty}$

$= \dfrac{A^2\vec{x}^{(k-2)}}{\|A^2\vec{x}^{(k-2)}\|_\infty} = \cdots = \dfrac{A^k\vec{x}^{(0)}}{\|A^k\vec{x}^{(0)}\|_\infty}$

$$= \frac{a_1 \lambda_1^k \left[\vec{x}_1 + \sum_{j=2}^{n} \frac{a_j}{a_1} \left(\frac{\lambda_j}{\lambda_1}\right)^k \vec{x}_j \right]}{\left\| a_1 \lambda_1^k \left[\vec{x}_1 + \sum_{j=2}^{n} \frac{a_j}{a_1} \left(\frac{\lambda_j}{\lambda_1}\right)^k \vec{x}_j \right] \right\|_\infty} \approx \underbrace{\frac{a_1}{|a_1|} \frac{\vec{x}_1}{\|\vec{x}_1\|_\infty} \frac{\lambda_1^k}{|\lambda_1|^k}}_{\vec{v}}$$

when k is big

$\left(\because \left|\frac{\lambda_j}{\lambda_1}\right| < 1 \text{ for } j = 2, \dots, n \right)$

Note: $\vec{v}$ is an eigenvector associated to λ_1.

In fact,

$$\|A\vec{x}^{(k)}\|_\infty \longrightarrow \left\| A \left(\frac{a_1}{|a_1|} \frac{\vec{x}_1}{\|\vec{x}_1\|_\infty} \right) \right\|_\infty = \left\| \frac{a_1}{|a_1|} \lambda_1 \frac{\vec{x}_1}{\|\vec{x}_1\|_\infty} \right\|_\infty = |\lambda_1|$$

$\frac{\lambda_1^k}{|\lambda_1|^k}$ can be removed under $\|\cdot\|_\infty$

Power method:

Step 1: Initialize $\vec{x}^{(0)}$ (such that $\vec{x}^{(0)} = a_1 \vec{x}_1 + \cdots + a_n \vec{x}_n$ with $a_1 \neq 0$)

Step 2: Compute $\vec{x}^{(k+1)} = A\vec{x}^{(k)} / \|A\vec{x}^{(k)}\|_\infty$

Step 3: Compute $\lambda^{(k+1)} = \|A\vec{x}^{(k+1)}\|_\infty$

Then: $\lambda^{(k)} \longrightarrow |\lambda_1|$ as $k \to \infty$

How about if A is NOT diagonalizable?

(Jordan Canonical Form)

Let $A = VJV^{-1}$, J = Jordan Canonical Form

Since the dominant eigenvalue, λ_1, has multiplicity 1, the first Jordan block of A must be a 1×1 matrix.

$$\therefore \quad J = \begin{pmatrix} \lambda_1 & & & \\ & [J(\lambda_{i_2})] & & \\ & & \ddots & \\ & & & [J(\lambda_{i_k})] \end{pmatrix}$$

Recall:

$$J(\lambda_j) = \begin{pmatrix} \lambda_j & 1 & & 0 \\ & \lambda_j & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_j \end{pmatrix}$$

Assume $c_1 \neq 0$, then:

$$\vec{x}^{(k)} = \frac{A^k \vec{x}^{(0)}}{\| A^k \vec{x}^{(0)} \|_\infty} = \frac{(VJV^{-1})^k \vec{x}^{(0)}}{\| (VJV^{-1})^k \vec{x}^{(0)} \|_\infty}$$

$$= \frac{(VJ^kV^{-1})(c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n)}{\| (VJ^kV^{-1})(c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n) \|_\infty}$$

$$= \frac{VJ^k(c_1\vec{e}_1 + c_2\vec{e}_2 + \dots + c_n\vec{e}_n)}{\| VJ^k(c_1\vec{e}_1 + c_2\vec{e}_2 + \dots + c_n\vec{e}_n) \|_\infty} \qquad \left(\because V^{-1}\vec{v}_j = \vec{e}_j = \begin{array}{c} \\ \\ \\ j^{th} \rightarrow \\ \\ \\ \end{array} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right)$$

$$= \frac{c_1\lambda_1^k\left(\vec{v}_1 + \frac{1}{c_1}V\left(\frac{1}{\lambda_1}J\right)^k(c_2\vec{e}_2 + \dots + c_n\vec{e}_n)\right)}{\| c_1\lambda_1^k\left(\vec{v}_1 + \frac{1}{c_1}V\left(\frac{1}{\lambda_1}J\right)^k(c_2\vec{e}_2 + \dots + c_n\vec{e}_n)\right) \|_\infty} \qquad \left(\text{Note: } J^k\vec{e}_1 = \lambda_1^k\vec{e}_1 \Rightarrow VJ^k\vec{e}_1 = \lambda_1^k\vec{v}_1 \right)$$

Now, $$\left(\frac{1}{\lambda_1}J\right)^k = \begin{pmatrix} 1 & & & \\ & \left[\frac{1}{\lambda_1}J(\lambda_{i_2})\right]^k & & \\ & & \ddots & \\ & & & \left[\frac{1}{\lambda_1}J(\lambda_{i_k})\right]^k \end{pmatrix} \to \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$$

as $k \to \infty$

$$\left|\frac{\lambda_{i_2}}{\lambda_1}\right| < 1 \qquad \left|\frac{\lambda_{i_k}}{\lambda_1}\right| < 1$$

$$\therefore \quad \vec{x}^{(k)} \approx \frac{c_1 \lambda_1^k \vec{v}_1}{\|c_1 \lambda_1^k \vec{v}_1\|_\infty}$$ when k is large.

$$\therefore \quad \|A\vec{x}^{(k)}\|_\infty \to |\lambda_1|$$ as $k \to \infty$ as before!

<u>Example</u>: Consider: $A = \begin{pmatrix} 0 & 11 & -5 \\ -2 & 17 & -7 \\ -4 & 26 & 10 \end{pmatrix}$. Using power method with $\vec{x}^{(0)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, find the spectral radius of A.

<u>Solution</u>: $\vec{x}^{(1)} = \dfrac{A\vec{x}^{(0)}}{\|A\vec{x}^{(0)}\|_\infty} = \dfrac{(6,\ 8,\ 12)^T}{\|(6,\ 8,\ 12)^T\|_\infty} = \begin{pmatrix} \frac{1}{2} \\ \frac{2}{3} \\ 1 \end{pmatrix}$.

$\vec{x}^{(2)} = \dfrac{A\vec{x}^{(1)}}{\|A\vec{x}^{(1)}\|_\infty} = \dfrac{(7/3,\ 10/3,\ 16/3)^T}{\|(7/3,\ 10/3,\ 16/3)^T\|_\infty} = \begin{pmatrix} 7/16 \\ 5/8 \\ 1 \end{pmatrix}$.

Compute $\|A\vec{x}^{(k)}\|_\infty$, we have:

k	6	8	10
$\|A\vec{x}^{(k)}\|_\infty$	4.025316	4.006270	4.001564

. The dominant eigenvalue ≈ 4

Generalization of Power method

Consider an invertible matrix A. Suppose A has eigenvalues:

$|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n| \; (>0)$

Consider A^{-1} (exist as all eigenvalues are non-zero). Then A^{-1} has eigenvalues:

$\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}$ with $\left|\frac{1}{\lambda_n}\right| > \left|\frac{1}{\lambda_{n-1}}\right| > \cdots > \left|\frac{1}{\lambda_1}\right|$

Extension: Apply Power's method on A^{-1} to obtain $\left|\frac{1}{\lambda_n}\right|$.

$\therefore$ the minimal eigenvalue can be determined! (Inverse Power method)

Remark: Computing A^{-1} is difficult! We solve: $A\vec{y} = \vec{x}^{(n)}$ in each iteration to determine $A^{-1}\vec{x}^{(n)}$.

Finding A^{-1} is equivalent to solving:

$A\vec{y} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \; A\vec{y} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \; A\vec{y} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$

$$A \underbrace{\begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{pmatrix}}_{A^{-1}} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & 0 & & \vdots \\ 0 & \vdots & & \vdots \\ 0 & 0 & & 1 \end{pmatrix}$$

Algorithm: (Inverse Power method)

Step 1: Pick $\vec{x}^{(0)}$ with $\|\vec{x}^{(0)}\|_\infty = 1$

Step 2: For $k = 1, 2, \ldots,$ solve $A\vec{w} = \vec{x}^{(k-1)}$.

Let $\vec{x}^{(k)} = \dfrac{\vec{w}}{\|\vec{w}\|_\infty}$.

Let $\rho_k = \|A\vec{x}^{(k)}\|_\infty$.

Remark: Again, $\rho_k \to |\lambda_n|$ as $k \to \infty$. ($\vec{x}^{(k)} \approx$ eigenvector of eigenvalue λ_n)