Lecture 16:

Another condition for the convergence (useful for analyzing SOR)

Let $A = N - P$ (N is invertible)

Iterative scheme : $N\vec{x}^{k+1} = P\vec{x}^{k} + \vec{b}$

Theorem: (Household - John) Suppose A and $(N^* + N - A)$ are self-adjoint positive-definite matrices, then the iterative scheme converges.

Example: Let $A = \begin{pmatrix} \alpha_1 & \beta_1 & & & 0 \\ \beta_1 & \alpha_2 & \beta_2 & & \\ & \beta_2 & \ddots & \ddots & \\ & & \ddots & \alpha_{n-1} & \beta_{n-1} \\ 0 & & & \beta_{n-1} & \alpha_n \end{pmatrix}$ be real symmetric tridiagonal matrix. Suppose A is positive-definite. Prove that the Gauss-Seidel method converges.

Solution: For Gauss-Seidel method,

$$N = \begin{pmatrix} \alpha_1 & & 0 \\ \beta_1 & \alpha_2 & \\ & \ddots & \ddots \\ 0 & \beta_{n-1} & \alpha_n \end{pmatrix} \quad \text{and} \quad N^* + N - A = \begin{pmatrix} \alpha_1 & & & 0 \\ & \alpha_2 & & \\ & & \ddots & \\ & & & \alpha_{n-1} & \\ 0 & & & & \alpha_n \end{pmatrix}$$

Then, $N^* + N - A$ is symmetric.

Also, $\overset{\vec{e}_i^{\,T}}{(0,\dots,\underset{\overset{\uparrow}{i\text{th}}}{1},0\cdots 0)}\; A\; \underset{\vec{e}_i}{\begin{pmatrix}0\\ \vdots\\ 1\\ 0\\ \vdots\\ 0\end{pmatrix}} \leftarrow i\text{th} = \alpha_i > 0$. $\therefore$ $N + N^* - A$ is positive definite

($\because$ all eigenvalues are positive)

By Householder - John Theorem, Gauss-Seidel method converges.

Example: Suppose A is (real) self-adjoint positive-definite. Using Householder-John theorem, prove that: SOR method converges if and only if $0 < \omega < 2$.

Solution: Note that $N_{SOR} = L + \frac{1}{\omega} D$. Now, $L = U^*$ (A is self-adjoint)

$$\therefore \quad N_{SOR} + N_{SOR}^* - A = \left(\frac{2}{\omega} - 1\right) D$$

$\therefore$ $N_{SOR} + N_{SOR}^* - A$ is also self-adjoint positive-definite if $0 < \omega < 2$.

$\therefore$ By Householder - John theorem, SOR converges.

Important example:

Consider an nxn tridiagonal matrix of the form:

$$A = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{pmatrix}$$

We want to study the convergence of Jacobi and SOR methods to solve $A\vec{x} = \vec{b}$.

In fact, for any nxn matrix of the form:

$$T_\alpha = \begin{pmatrix} \alpha & -1 & & & \\ -1 & \alpha & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & \alpha \end{pmatrix},$$

its eigenvectors are given by:

$$\vec{q}_j = \begin{pmatrix} \sin(j\theta) \\ \sin(2j\theta) \\ \vdots \\ \sin(nj\theta) \end{pmatrix}$$

w/ eigenvalue

$$\lambda_j = \alpha - 2\cos(j\theta)$$

$$\left(j = 1, 2, \ldots, n-1, \quad \theta = \frac{\pi}{n+1}\right)$$

Proof:

$$T_\alpha \vec{q}_j = \begin{pmatrix} \alpha \sin j\theta - \sin(2j\theta) \\ \alpha \sin(2j\theta) - \sin(j\theta) - \sin(3j\theta) \\ \alpha \sin(3j\theta) - \sin(2j\theta) - \sin(4j\theta) \\ \vdots \\ \alpha \sin(nj\theta) - \sin((n-1)j\theta) \end{pmatrix}$$

Using trigonometric formula:

$$\sin(a+b) + \sin(a-b) = 2\sin a \cos b$$

$$\therefore \quad T_\alpha \vec{q}_j = \begin{pmatrix} \alpha \sin j\theta - 2\sin j\theta \cos j\theta \\ \alpha \sin(2j\theta) - 2\cos j\theta \sin 2j\theta \\ \vdots \\ \alpha \sin(nj\theta) - 2\cos(j\theta)\sin(nj\theta) \end{pmatrix} = \underbrace{(\alpha - 2\cos(j\theta))}_{\lambda_j} \vec{q}_j$$

Let $A = L + D + U$. Then:

$$\alpha D^{-1} L + \frac{1}{\alpha} D^{-1} U = \begin{pmatrix} 0 & -\frac{1}{2\alpha} & & \\ -\frac{\alpha}{2} & 0 & \ddots & \\ & \ddots & \ddots & -\frac{1}{2\alpha} \\ & & -\frac{\alpha}{2} & 0 \end{pmatrix}$$

which is tridiagonal.

Hence, it is consistently ordered.

In fact, we can show that for any general matrix B of the form:

$$B = \begin{pmatrix} a & b & & \\ c & a & \ddots & \\ & \ddots & \ddots & b \\ & & c & a \end{pmatrix},$$

the eigenvalues are given by:

$$\mu_j = a - 2\sqrt{bc}\cos\left(\frac{j\pi}{n+1}\right)$$

(Out of Scope)

for $j = 1, 2, \ldots, n$

In our case, $\mu_j = -\cos\left(\frac{j\pi}{n+1}\right)$ for $j = 1, 2, .., n$ which are independent of α (Put $a = 0$, $b = -\frac{1}{2\alpha}$, $c = -\frac{\alpha}{2}$)

$\therefore$ $\alpha D^{-1} L + \frac{1}{\alpha} D^{-1} U$ is consistently ordered.

Now, $\rho(M_J) = \cos\left(\frac{\pi}{n+1}\right)$

$\therefore$ the optimal ω is:

$$\omega_{opt} = \frac{2}{1 + \sqrt{1 - \rho(M_J)^2}} = \frac{2}{1 + \sin\left(\frac{\pi}{n+1}\right)} \quad //$$