MATH 3310 Assignment 1
Due on February 5 before 6p.m.

1. Using integrating factor method solve:

   (a)
   $$x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 10x^2 + 5x + 1, \quad x > 0$$
   with condition $y(1) = c > 0$

   (b)
   $$-2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3y = 15x^2 + 18x - 11$$
   with condition $\frac{\mathrm{d}y}{\mathrm{d}x}(0) = 0, y(1) = 1$

2. Given a 2π-periodic function $f(x)$. Let $a_0, a_1, ..., a_N$ and $b_1, ..., b_N$ are chosen in such a way that it minimizes:
   $$E(a_0, ..., a_N, b_1, ..., b_N) = \int_0^{2\pi} [f(x) - (\sum_{j=0}^{N} a_j \cos jx + \sum_{j=1}^{N} b_j \sin jx)]^2 dx$$
   What are $a_0, a_1, ..., a_N$ and $b_1, ..., b_N$? Please show your answers with details.

3. In class, we saw that given $f(x)$ that is periodic over the interval $[0, 2\pi]$, we can compute the coefficients of its Fourier series by the following formulas:
   $$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\mathrm{d}x$$
   $$a_m = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos mx\mathrm{d}x$$
   $$b_m = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin mx\mathrm{d}x$$
   By using integration with substitution, or otherwise, prove that if $f(x)$ is periodic over an interval $[-L, L]$, then the above formulas shall be replaced

by:

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) \mathrm{d}x$$
$$a_m = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{m\pi x}{L}\right) \mathrm{d}x$$
$$b_m = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{m\pi x}{L}\right) \mathrm{d}x$$

Compute the Fourier series of $f(x) = c_1 x + c_2 |x|, -3 < x < 3$ with period 6.

4. Using the spectral method, solve the following problem:

$$u_t = 8u_{xx}, 0 < x < 4, t > 0$$
$$u(0,t) = u(4,t) = 0$$
$$u(x,0) = 5\sin(2\pi x) - 5\sin(5\pi x) + 10\sin(8\pi x)$$

5. Using the spectral method, solve the following problem:

$$u_t - u_{xx} = 2t\sin(nx) + t^2 \sin(mx), \quad 0 < x < 2\pi, \quad t > 0, \quad m > n > 0$$
$$u(0,t) = u(2\pi, t) = 0$$
$$u(x,0) = 2\sin(nx) + \sin(mx)$$

6. (a) Compute the Fourier transform of $f(x) = e^{-a|x|}$ for $a > 0$.

   (b) Solve the PDE:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad x \in (-\infty, \infty), \quad t > 0$$
$$u(x,0) = e^{-a|x|}, \text{ where } a > 0.$$

   Please express your answer in terms of convolution. (**Hint:** You may use the fact without proof that the inverse Fourier transform of $\hat{\phi}(k,t) = e^{-k^2 t}$ is $\phi(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$.)