Lecture 7: Recall:

Understanding convolution:

Discrete convolution:

$$\underset{g * I(n,m)}{\underset{\|}{v(n,m)}} = \underbrace{\sum_{n'=0}^{N-1} \sum_{m'=0}^{N-1} g(n-n', m-m')\, I(n',m')}_{\text{Linear combination of pixel values of } I}$$

In particular, if $g(k,l)$ is only non-zero around $(0,0)$, then, $g * I(n,m)$ is a linear combination of pixel value of I around (n,m)!!

DFT of convolution:

Recall: $$g * w(n,m) = \sum_{n'=0}^{N-1} \sum_{m'=0}^{N-1} g(n-n', m-m')\, w(n', m')$$

$\left(g, m \in M_{N \times M}(\mathbb{R}) \right)$

Then, the DFT of $g * w(p,q) = MN\, \mathrm{DFT}(g)(p,q)\, \mathrm{DFT}(w)(p,q)$

$\therefore$ DFT of convolution can be reduced to simple multiplication!

Recall: Shift-invariant image transformation = 2D convolution.

$\therefore$ Easy computation/manipulation of shift-invariant transf. after DFT!!

Image enhancement in the frequency domain:

Goal: 1. Remove $\underbrace{\text{high-frequency components}}_{\uparrow \text{ noise}}$ (low-pass filter) for image denoising

2. Remove low-frequency components (high-pass filter) for the extraction of $\underbrace{\text{image details}}_{\text{edges}}$

High/Low frequency components of $\hat{F} \leftarrow \mathrm{DFT}(F)$

Let F be a $N \times N$ image, $N =$ even. Let $\hat{F} = \mathrm{DFT}(F)$.

$$\therefore \underbrace{\hat{F}(k,l)}_{\text{Fourier coefficients of } F \text{ at } (k,l)} = \frac{1}{N^2} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} F(m,n)\, e^{-j2\pi\left(\frac{mk+nl}{N}\right)}$$

$$\boxed{\overline{e^{j\theta}} = e^{j(-\theta)}}$$

$j = \sqrt{-1}$, $\cos\theta - j\sin\theta$

Observe that: for $0 \le k, l \le \frac{N}{2} - 1$

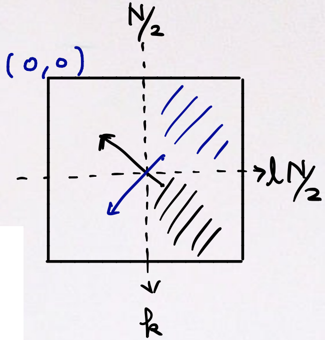
$$\hat{F}\left(\frac{N}{2}+k, \frac{N}{2}+l\right) = \frac{1}{N^2} \sum_m \sum_n F(m,n)\, \underbrace{e^{-j\frac{2\pi}{N}\left(m\left(\frac{N}{2}+k\right) + n\left(\frac{N}{2}+l\right)\right)}}$$

$$= \frac{1}{N^2} \sum_m \sum_n F(m,n)\, (-1)^{m+n}\, e^{-j\frac{2\pi}{N}(m(-k)+n(-l))}$$

$$= \overline{\frac{1}{N^2} \sum_m \sum_n F(m,n)\, e^{-j\frac{2\pi}{N}\left(m\left(\frac{N}{2}-k\right) + n\left(\frac{N}{2}-l\right)\right)}}$$

$$\hat{F}\left(\frac{N}{2}+k, \frac{N}{2}+l\right) = \overline{\hat{F}\left(\frac{N}{2}-k, \frac{N}{2}-l\right)}$$

$$F(m,n) = \sum_k \sum_l \hat{F}(k,l)\, e^{j\frac{2\pi}{N}(mk+nl)}$$



We have:

$$\begin{aligned}
F(m,n) = & \sum_{0\le k,l\le \frac{N}{2}-1} \left[\hat{F}\left(\frac{N}{2}+k, \frac{N}{2}+l\right) e^{j\frac{2\pi}{N}[(\underline{\frac{N}{2}+k})m+(\underline{\frac{N}{2}+l})n]}\right] \\
& + \sum_{1\le k,l\le \frac{N}{2}-1} \left[\overline{\hat{F}\left(\frac{N}{2}+k, \frac{N}{2}+l\right)} e^{j\frac{2\pi}{N}[(\underline{\frac{N}{2}-k})m+(\underline{\frac{N}{2}-l})n]}\right] \\
& + \sum_{0\le k,l\le \frac{N}{2}-1} \left[\hat{F}\left(\frac{N}{2}+k, \frac{N}{2}-l\right) e^{j\frac{2\pi}{N}[(\underline{\frac{N}{2}+k})m+(\underline{\frac{N}{2}-l})n]}\right] \\
& + \sum_{1\le k,l\le \frac{N}{2}-1} \left[\overline{\hat{F}\left(\frac{N}{2}+k, \frac{N}{2}-l\right)} e^{j\frac{2\pi}{N}[(\underline{\frac{N}{2}-k})m+(\underline{\frac{N}{2}+l})n]}\right] \\
& + \sum_{0\le l\le \frac{N}{2}-1} \overset{(-1)^m}{\hat{F}}\left(0, \frac{N}{2}+l\right) e^{j\frac{2\pi}{N}[(\frac{N}{2}+l)n]} + \sum_{1\le l\le \frac{N}{2}-1} \overline{\hat{F}\left(0, \frac{N}{2}+l\right)} e^{j\frac{2\pi}{N}[(\frac{N}{2}-l)n]} \quad (-1)^m \\
& + \sum_{0\le k\le \frac{N}{2}-1} \hat{F}\left(\frac{N}{2}+k, 0\right) \underbrace{e^{j\frac{2\pi}{N}[(\frac{N}{2}+k)m]}}_{(-1)^n} + \sum_{1\le k\le \frac{N}{2}-1} \overline{\hat{F}\left(\frac{N}{2}+k, 0\right)} \underbrace{e^{j\frac{2\pi}{N}[(\frac{N}{2}-k)m]}}_{(-1)^n} + \hat{F}(0,0)
\end{aligned}$$

Observation:

1. When k and l are close to $N/2$, $\hat{F}\left(\frac{N}{2}+k, \frac{N}{2}+l\right)$ is associated to $e^{-j\frac{2\pi}{N}\left(\left(\frac{N}{2}+k\right)m+\left(\frac{N}{2}+l\right)n\right)}$

   $k \ll N$, $l \ll N$

   $$\approx e^{-j\frac{2\pi}{N}(k'm+l'n)} \quad \text{where } (k',l') \approx (0,0)$$

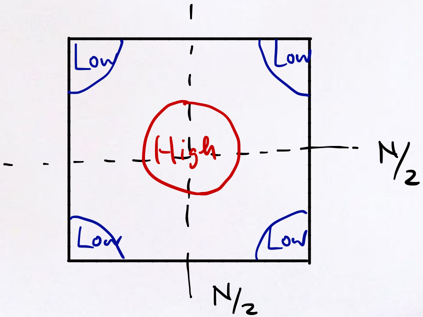
   $$= \cos\left(\frac{2\pi}{N}(k'm+l'n)\right) + i\sin\left(\frac{2\pi}{N}(k'm+l'n)\right)$$

   Low-frequency if $(k,l)\approx(0,0)$

   $\therefore$ Fourier coefficients at the bottom right are associated to low frequency components!

2. Similarly, we can check that Fourier coefficients at the 4 corners are associated to low frequency components.

3. Fourier coefficients in the middle are associated to high-frequency components



!!

$\therefore$ High-pass filtering

= Remove coefficients at 4 corners

Low-pass filtering

= Remove coefficients at the center

Centralisation:

Assume periodic conditions on F.

We can let $\tilde{F}(u,v) = \hat{F}\left(u-\frac{N}{2}, v-\frac{N}{2}\right)$ where $0 \leq u \leq N-1$

$0 \leq v \leq N-1$

Then, High-frequency components are located at 4 corners of $\tilde{F}(u,v)$

Low-frequency components are located at center of $\tilde{F}(u,v)$

Mathematically, Consider the discrete Fourier transform of $(-1)^{x+y}F(x,y)$:

$$= e^{j2\pi\frac{\left(\frac{N}{2}x+\frac{N}{2}y\right)}{N}}$$

$$DFT(F(x,y)(-1)^{x+y})(u,v)$$

$$=\frac{1}{N^2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)e^{j\pi(x+y)}exp\left(-j2\pi\left(\frac{ux}{N}+\frac{vy}{N}\right)\right)$$

$$=\frac{1}{N^2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)exp\left(-j2\pi\left(\frac{(u-N/2)x}{\cancel{M}\ N}+\frac{(v-N/2)y}{N}\right)\right)$$

$$=\hat{F}\left(u-\frac{N}{2}, v-\frac{N}{2}\right) = \tilde{F}(u. \quad , v \quad .)$$

Therefore, to compute $\tilde{F}(u,v)$, we can compute DFT of $(-1)^{x+y}f(x,y)$.

Definition 3.5: A **low-pass filter (LPF)** (LPF) leaves low frequencies unchanged, while attenuating the high frequencies.

A **high-pass filter (HPF)** leaves high frequencies unchanged, while attenuating the low frequencies.

Basic steps of filtering in the frequency domain

1. Multiply $f(x, y)$ by $(-1)^{x+y}$.
2. Compute $\tilde{F}(u, v) = DFT(f(x, y)(-1)^{x+y})(u, v)$.
3. Multiply $\tilde{F}$ by a real "filter" function $H(u, v)$ to get

$$G(u, v) = H(u, v)\tilde{F}(u, v)$$

   (point-wise multiplication, but not matrix multiplication)

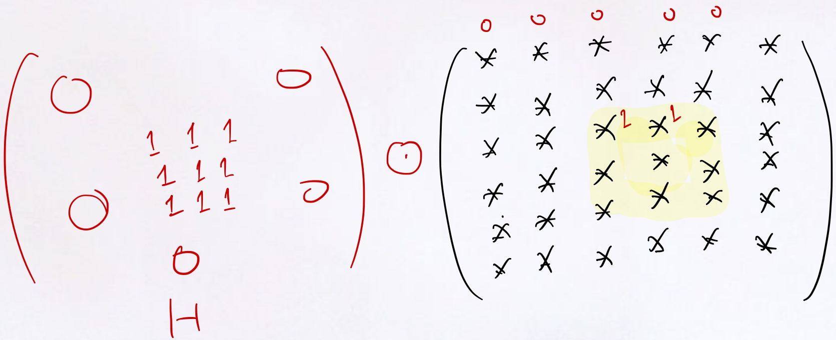
4. Compute inverse DFT of $G(u, v)$.
5. Take real part of the result in Step 4.
6. Multiply the result in Step 5 by $(-1)^{x+y}$.

Remark: 1. H is taken either to remove high-frequency coefficients/low-frequency coefficients.

2. $\mathcal{F}^{-1}(G(u,v)) = g(x,y) = (1/N)\,\mathcal{F}^{-1}(H(u,v)) * \mathcal{F}^{-1}(\tilde{F}(u,v)) = (1/N^2)\, h * \tilde{f}(x,y)$

Filtering in the frequency domain = Linear filtering in the spatial domain!

$$\mathcal{F}(h * \tilde{f}) = N^2\, \mathcal{F}(H) \odot \mathcal{F}(f)$$



Example of Low-pass filters for image denoising

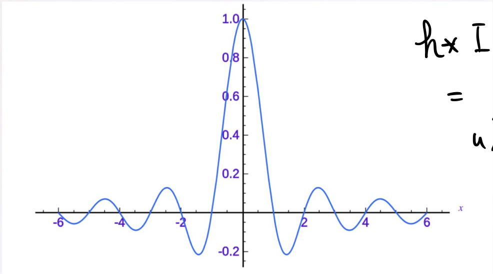
Assume that we work on the centered spectrum!

That is, consider $\hat{F}(u,v)$ where $-N/2 \le u \le N/2 - 1$, $-N/2 \le v \le N/2 - 1$.

1 Ideal low pass filter (ILPF):

$$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) := u^2 + v^2 \le D_0^2 \\ 0 & \text{if } D(u,v) > D_0^2 \end{cases}$$

In 1-dim cross-section, $\mathcal{F}^{-1}(H(u,v))$ looks like:



$$h * I(x,y) = \sum_{u,v} h(x-u, y-v)\, I(u,v)$$

every pixel values of I has an effect on $h * I(x,y)$!!

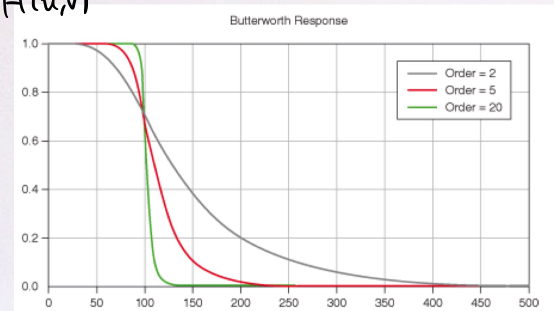
Good: Simple

Bad : Produce ringing effect!

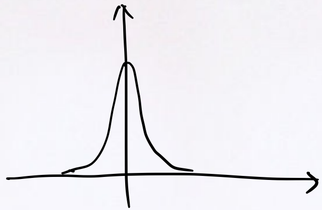
2. Butterworth low-pass filter (BLPF) of order n ($n \geq 1$ integer) :

$$H(u,v) = \frac{1}{1 + \left(D(u,v)/D_0\right)^n}$$

$H(u,v)$ in 1-dim



$\mathcal{F}^{-1}(H(u,v))$ in 1-dim

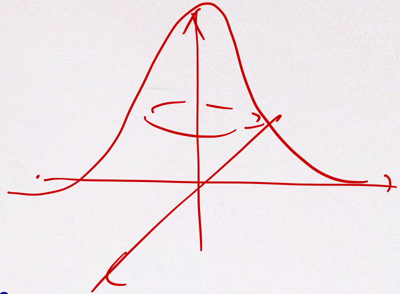


Good: Produce less / no visible ringing effect if n is carefully chosen!!

3. Gaussian low-pass filter

$$H(u,v) = \exp\left(-\frac{\overset{u^2+v^2}{D(u,v)}}{2\sigma^2}\right)$$

σ = spread of the Gaussian function



F.T. of Gaussian is also Gaussian!!

Good: No visible ringing effect!!

Examples for high-pass filtering for feature extraction

1. Ideal high-pass filter: (IHPF)

$$H(u,v) = \begin{cases} 0 & \text{if } D(u,v) \le D_0^2 \\ 1 & \text{if } D(u,v) > D_0^2 \end{cases}$$

Bad: Produce ringing

2. Butterworth high-pass filter:

$$H(u,v) = \frac{1}{1 + \left(\frac{D_0}{D(u,v)}\right)^n}$$

$(H(u,v) = 0 \text{ if } D(u,v) = 0)$

Choose the right n

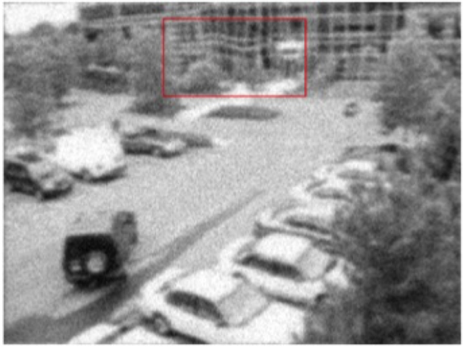
Good: Less ringing

3. Gaussian high-pass filter

$$H(u,v) = 1 - e^{-\left(\frac{D(u,v)}{2\sigma^2}\right)}$$

Good: No visible ringing!

Image deblurring



Atmospheric turbulence

Motion Blur

Speeding problem

Image deblurring in the frequency domain:

Mathematical formulation of image blurring

Let g be the observed (blurry) image.

Let f be the original (good) image.

Model g as: $g = H(f) + n$

where H is the degradation function/operator and n is the additive noise.

Assumption on H:

1. H is position invariant:

   Let $g(x,y) = H(f)(x,y)$ and let $\tilde{f}(x,y) := f(x-\alpha, y-\beta)$.

   Then: $H(\tilde{f})(x,y) = g(x-\alpha, y-\beta)$

2. Linear: $H(f_1 + f_2) = H(f_1) + H(f_2)$

   $H(\alpha f) = \alpha H(f)$ where α is a scalar multiplication.

3. Linearity can be extended to integral:

$$H\left(\iint \alpha(u,v) f(x-u, y-v)\, du\, dv\right) = \iint \alpha(u,v) H(f)(x-u, y-v)\, du\, dv$$

With the above assumption, consider an impluse signal:

$$\delta(x,y) = \begin{cases} 1 & \text{if } (x,y) = (0,0) \\ 0 & \text{if } (x,y) \neq (0,0) \end{cases}$$

Then: $f(x,y) = f * \delta(x,y) = \displaystyle\sum_{\alpha=-M/2}^{M/2-1} \sum_{\beta=-N/2}^{N/2-1} f(\alpha,\beta)\,\delta(x-\alpha, y-\beta)$

$$\therefore \quad g(x,y) = H(f)(x,y)$$

$$= \sum_{\alpha=-M/2}^{M/2-1} \sum_{\beta=-N/2}^{N/2-1} f(\alpha,\beta)\, H(\delta)(x-\alpha, y-\beta) \quad \text{(by linearity and position-invariant)}$$

$$= \sum_{\alpha=-M/2}^{M/2-1} \sum_{\beta=-N/2}^{N/2-1} f(\alpha,\beta)\, h(x-\alpha, y-\beta) \quad \text{where } h(x,y) = H(\delta)(x,y)$$

$$= f * h(x,y)$$

$\therefore$ With the above assumption,

Degradation / Blur = Convolution