Lecture 2:

Representation of $\mathcal{O}$ by a matrix H:

We can write:

$$
\left.
\begin{aligned}
g(\alpha,\beta) = {} & f(1,1)\,h(1,\alpha,1,\beta) + f(2,1)\,h(2,\alpha,1,\beta) + \dots + f(N,1)\,h(N,\alpha,1,\beta) \\
& + f(1,2)\,h(1,\alpha,2,\beta) + \dots + f(N,2)\,h(N,\alpha,2,\beta) \\
& \dots \\
& + f(1,N)\,h(1,\alpha,N,\beta) + \dots + f(N,N)\,h(N,\alpha,N,\beta)
\end{aligned}
\right\}
$$

Each (α,β) is associated to a linear equations.

Arrange:

$$
\vec{f} = \begin{pmatrix} f(1,1) \\ \vdots \\ f(N,1) \\ f(1,2) \\ \vdots \\ f(N,2) \\ \vdots \\ f(1,N) \\ \vdots \\ f(N,N) \end{pmatrix} ; \quad
\vec{g} = \begin{pmatrix} g(1,1) \\ g(2,1) \\ \vdots \\ g(N,1) \\ g(1,2) \\ \vdots \\ g(N,2) \\ \vdots \\ g(1,N) \\ g(2,N) \\ \vdots \\ g(N,N) \end{pmatrix} .
$$

In matrix form, let

$$
\vec{g} = \begin{pmatrix} g(1,1) \\ \vdots \\ g(N,1) \\ \vdots \\ g(1,N) \\ \vdots \\ g(N,N) \end{pmatrix} .
$$

Then: $\vec{g} = \underset{N^2 \times N^2 \text{ matrix}}{H}\,\vec{f}$

By careful examination, we see that:

$$
\begin{array}{cc}
 & y \longrightarrow \\
\beta \Big\downarrow & H = \begin{pmatrix}
\begin{pmatrix} & x \rightarrow \\ \alpha \downarrow & \begin{pmatrix} y = 1 \\ \beta = 1 \end{pmatrix} \end{pmatrix} & \begin{pmatrix} & x \rightarrow \\ \alpha \downarrow & \begin{pmatrix} y = 2 \\ \beta = 1 \end{pmatrix} \end{pmatrix} & \cdots & \begin{pmatrix} & x \rightarrow \\ \alpha \downarrow & \begin{pmatrix} y = N \\ \beta = 1 \end{pmatrix} \end{pmatrix} \\
\begin{pmatrix} & x \rightarrow \\ \alpha \downarrow & \begin{pmatrix} y = 1 \\ \beta = 2 \end{pmatrix} \end{pmatrix} & \begin{pmatrix} & x \rightarrow \\ \alpha \downarrow & \begin{pmatrix} y = 2 \\ \beta = 2 \end{pmatrix} \end{pmatrix} & \cdots & \begin{pmatrix} & x \rightarrow \\ \alpha \downarrow & \begin{pmatrix} y = N \\ \beta = 2 \end{pmatrix} \end{pmatrix} \\
\vdots & \vdots & & \vdots \\
\begin{pmatrix} & x \rightarrow \\ \alpha \downarrow & \begin{pmatrix} y = 1 \\ \beta = N \end{pmatrix} \end{pmatrix} & \begin{pmatrix} & x \rightarrow \\ \alpha \downarrow & \begin{pmatrix} y = 2 \\ \beta = N \end{pmatrix} \end{pmatrix} & \cdots & \begin{pmatrix} & x \rightarrow \\ \alpha \downarrow & \begin{pmatrix} y = N \\ \beta = N \end{pmatrix} \end{pmatrix}
\end{pmatrix}
\end{array}
$$

Meaning of

$$
\begin{pmatrix} & x \rightarrow \\ \alpha \downarrow & \begin{pmatrix} y = i \\ \beta = j \end{pmatrix} \end{pmatrix} :
\begin{pmatrix}
h(1,1,i,j) & h(2,1,i,j) & \cdots & h(N,1,i,j) \\
h(1,2,i,j) & h(2,2,i,j) & \cdots & h(N,2,i,j) \\
\vdots & \vdots & & \vdots \\
h(1,N,i,j) & h(2,N,i,j) & \cdots & h(N,N,i,j)
\end{pmatrix} \in M_{N \times N}
$$

$$
h(\overset{\substack{\text{col}\\ \text{of small}\\ \text{block}}}{x}, \overset{\substack{\text{row}\\ \text{of small}\\ \text{block}}}{\alpha}, \overset{\substack{\text{col}\\ \text{of block}\\ \text{matrix}}}{y}, \overset{\substack{\text{row of}\\ \text{block matrix}}}{\beta})
$$

<u>Definition</u>: H is called the <u>transformation matrix</u> of $\mathcal{O}$.

Example 1.1 A linear operator is such that it replaces the value of each pixel by the average of its four nearest neighbours. Assume the image is repeated in all directions. Apply this operator $\mathcal{O}$ to a 3×3 image. Find the transformation matrix corresponding to $\mathcal{O}$.

Solution:

$$
3 \times 3 \text{ image} = \begin{array}{ccc}
 & \begin{array}{ccc} f_{31} & f_{32} & f_{33} \end{array} & \\
\begin{array}{c} f_{13} \\ f_{23} \\ f_{33} \end{array} & \begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{pmatrix} & \begin{array}{c} f_{11} \\ f_{21} \\ f_{31} \end{array} \\
 & \begin{array}{ccc} f_{11} & f_{12} & f_{13} \end{array} &
\end{array}
$$

$$
g_{22} = \frac{f_{12} + f_{21} + f_{23} + f_{32}}{4} \quad ; \quad g_{33} = \frac{f_{23} + f_{32} + f_{31} + f_{13}}{4}
$$

etc ...

By careful examination, we see that

$$\begin{bmatrix} 0 & 1/4 & 1/4 & 1/4 & 0 & 0 & 1/4 & 0 & 0 \\ 1/4 & 0 & 1/4 & 0 & 1/4 & 0 & 0 & 1/4 & 0 \\ 1/4 & 1/4 & 0 & 0 & 0 & 1/4 & 0 & 0 & 1/4 \\ 1/4 & 0 & 0 & 0 & 1/4 & 1/4 & 1/4 & 0 & 0 \\ 0 & 1/4 & 0 & 1/4 & 0 & 1/4 & 0 & 1/4 & 0 \\ 0 & 0 & 1/4 & 1/4 & 1/4 & 0 & 0 & 0 & 1/4 \\ 1/4 & 0 & 0 & 1/4 & 0 & 0 & 0 & 1/4 & 1/4 \\ 0 & 1/4 & 0 & 0 & 1/4 & 0 & 1/4 & 0 & 1/4 \\ 0 & 0 & 1/4 & 0 & 0 & 1/4 & 1/4 & 1/4 & 0 \end{bmatrix}$$

What is $h(2,3,2,1)$?

What is $h(1,2,3,3)$?

$h(2,3,2,1) = 0$

$h(1,2,3,3) = 1/4$.

Recall:

$$H = \begin{pmatrix} \begin{pmatrix} & x \to \\ \alpha\downarrow & \begin{pmatrix} y=1 \\ \beta=1 \end{pmatrix} \end{pmatrix} & \begin{pmatrix} & x \to \\ \alpha\downarrow & \begin{pmatrix} y=2 \\ \beta=1 \end{pmatrix} \end{pmatrix} & \cdots & \begin{pmatrix} & x \to \\ \alpha\downarrow & \begin{pmatrix} y=N \\ \beta=1 \end{pmatrix} \end{pmatrix} \\ \begin{pmatrix} & x \to \\ \alpha\downarrow & \begin{pmatrix} y=1 \\ \beta=2 \end{pmatrix} \end{pmatrix} & \begin{pmatrix} & x \to \\ \alpha\downarrow & \begin{pmatrix} y=2 \\ \beta=2 \end{pmatrix} \end{pmatrix} & \cdots & \begin{pmatrix} & x \to \\ \alpha\downarrow & \begin{pmatrix} y=N \\ \beta=2 \end{pmatrix} \end{pmatrix} \\ \vdots & \vdots & & \vdots \\ \begin{pmatrix} & x \to \\ \alpha\downarrow & \begin{pmatrix} y=1 \\ \beta=N \end{pmatrix} \end{pmatrix} & \begin{pmatrix} & x \to \\ \alpha\downarrow & \begin{pmatrix} y=2 \\ \beta=N \end{pmatrix} \end{pmatrix} & \cdots & \begin{pmatrix} & x \to \\ \alpha\downarrow & \begin{pmatrix} y=N \\ \beta=N \end{pmatrix} \end{pmatrix} \end{pmatrix}$$

Example 1.2 Consider an image transformation on a 2×2 image. Suppose the matrix representation of the image transformation is given by:

$$H = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 4 & 2 & 2 & 1 \\ 3 & 0 & 4 & 0 \\ 6 & 3 & 8 & 4 \end{pmatrix}.$$

Prove that the image transformation is separable. Find g_1 and g_2 such that:

$$h(x, \alpha, y, \beta) = g_1(x, \alpha) g_2(y, \beta).$$

Solution: H for a 2×2 image :

$$\begin{pmatrix} \begin{array}{cc} & x \to \\ \alpha\downarrow & \begin{pmatrix} y=1 \\ \beta=1 \end{pmatrix} \end{array} & \begin{array}{cc} & x \to \\ \alpha\downarrow & \begin{pmatrix} y=2 \\ \beta=1 \end{pmatrix} \end{array} \\ \begin{array}{cc} & x \to \\ \alpha\downarrow & \begin{pmatrix} y=1 \\ \beta=2 \end{pmatrix} \end{array} & \begin{array}{cc} & x \to \\ \alpha\downarrow & \begin{pmatrix} y=2 \\ \beta=2 \end{pmatrix} \end{array} \end{pmatrix} \in M_{4\times 4}$$

H is separable $\Leftrightarrow$ $h(x,\alpha,y,\beta) = g_1(x,\alpha)\, g_2(y,\beta)$.

Easy to check:

if H is separable: $H = \begin{pmatrix} g_2(1,1)\, G_1 & g_2(2,1)\, G_1 \\ g_2(1,2)\, G_1 & g_2(2,2)\, G_1 \end{pmatrix}$; $G_1 = \begin{pmatrix} g_1(1,1) & g_1(2,1) \\ g_1(1,2) & g_1(2,2) \end{pmatrix}$

$$\begin{pmatrix} \begin{pmatrix} h(1,1,1,1) & h(2,1,1,1) \\ h(1,2,1,1) & h(2,2,1,1) \end{pmatrix} & \begin{pmatrix} h(1,1,2,1) & h(2,1,2,1) \\ h(1,2,2,1) & h(2,2,2,1) \end{pmatrix} \\ \begin{pmatrix} \quad \end{pmatrix} & \begin{pmatrix} \quad \end{pmatrix} \end{pmatrix}$$

$$\begin{pmatrix} g_1(1,1)\,g_2(1,1) & g_1(2,1)\,g_2(1,1) \\ g_1(1,2)\,g_2(1,1) & g_1(2,2)\,g_2(1,1) \end{pmatrix} = g_2(1,1) \underbrace{\begin{pmatrix} g_1(1,1) & g_1(2,1) \\ g_1(1,2) & g_2(1,1) \end{pmatrix}}_{G_1}$$

$$g_2(2,1)\,G_1$$

In our case, $H = \begin{pmatrix} 2G_1 & 1G_1 \\ 3G_1 & 4G_1 \end{pmatrix}$; $G_1 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

$\therefore$

$g_1(1,1) = 1$	$g_2(1,1) = 2$
$g_1(2,1) = 0$	$g_2(2,1) = 1$
$g_1(1,2) = 2$	$g_2(1,2) = 3$
$g_1(2,2) = 1$ //	$g_2(2,2) = 4$ //

More about convolution

"Geometric" intrepretation of discrete convolution

Let $\quad f * g(\alpha, \beta) = \displaystyle\sum_{x=1}^{N} \sum_{y=1}^{N} f(x, y)\, g(\alpha - x, \beta - y)$

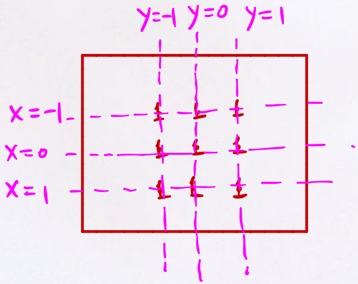
Consider a simple case where only several entries of g are non-zero.

Namely,

$g(0,0) = g(N,N) = 1 \;;\; g(1,0) = g(1,N) = 1 \;;\; g(-1,0) = g(N-1,N) = 1$

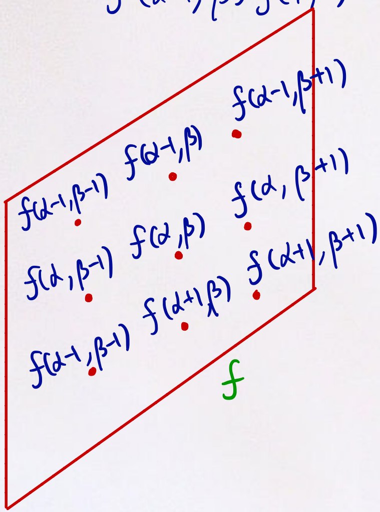
$g(0,1) = g(N,1) = 1 \;;\; g(1,1) = g(1,1) = 1 \;;\; g(-1,1) = g(N-1,1) = 1$

$g(0,-1) = g(N,N-1) = 1 \;;\; g(1,-1) = g(1,N-1) = 1 \;;\; g(-1,-1) = g(N-1,N-1) = 1$

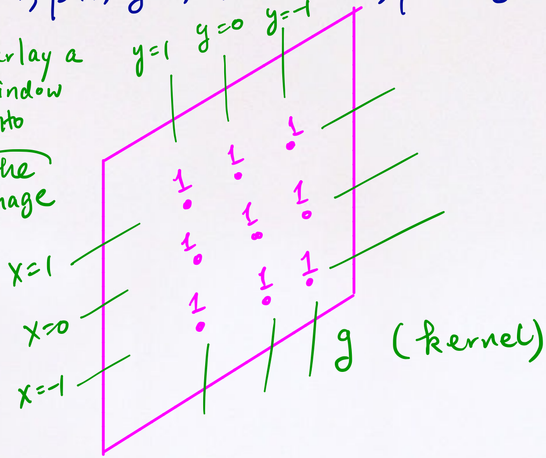


Expand the summation:

$$
\begin{aligned}
f * g(\alpha,\beta) = {} & f(\alpha,\beta)\, g(0,0) + f(\alpha,\beta+1)\, g(0,-1) + f(\alpha,\beta-1)\, g(0,1) + \\
& f(\alpha+1,\beta)\, g(-1,0) + f(\alpha+1,\beta+1)\, g(-1,-1) + f(\alpha+1,\beta-1)\, g(-1,1) + \\
& f(\alpha-1,\beta)\, g(1,0) + f(\alpha-1,\beta+1)\, g(1,-1) + f(\alpha-1,\beta-1)\, g(1,1) .
\end{aligned}
$$



Overlay a window onto the image



Properties of shift-invariant/separable image transformation

Definition: (Circulant matrix)

A circulant matrix $V := \mathrm{circ}(\vec{v})$ associated to a vector $\vec{v} = (v_0, v_1, \dots, v_{n-1})^T \in \mathbb{C}^n$ is a $n \times n$ matrix whose columns are given by iterations of shift operator T acting on $\vec{v}$. Here, $T : \mathbb{C}^n \to \mathbb{C}^n$ by

$$T\begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{n-1} \end{pmatrix} = \begin{pmatrix} v_{n-1} \\ v_0 \\ v_1 \\ \vdots \\ v_{n-2} \end{pmatrix}.$$

$\therefore$ k^{th} column is given by $T^{k-1}(\vec{v})$ $(k = 1, 2, \dots, n)$

$$\therefore \quad V = \begin{pmatrix} v_0 & v_{n-1} & v_{n-2} & \cdots & v_1 \\ v_1 & v_0 & v_{n-1} & \cdots & v_2 \\ \vdots & \vdots & v_0 & & \vdots \\ & & \vdots & & v_{n-1} \\ v_{n-1} & v_{n-2} & v_{n-3} & \cdots & v_0 \end{pmatrix}$$

Definition: (Block circulant)

V is block-circulant $\Leftrightarrow$ $V = \begin{pmatrix} H_0 & H_{n-1} & \cdots & H_1 \\ H_1 & H_0 & & H_2 \\ \vdots & \vdots & \cdots & \vdots \\ & & & H_{n-1} \\ H_{n-1} & H_{n-2} & \cdots & H_0 \end{pmatrix} \ni$

each H_i is a circulant matrix.

Theorem: If H = transf. matrix of shift-invariant operator,

then $H = \begin{pmatrix} A_{11} & \cdots & A_{1N} \\ A_{21} & \cdots & A_{2N} \\ \vdots & \ddots & \vdots \\ A_{N1} & \cdots & A_{NN} \end{pmatrix}$ where each A_{ij} is a circulant matrix.

(Assuming $h(x, \alpha, y, \beta) = g(\alpha - x, \beta - y)$ and g is periodic in 1st and 2nd argument)

Proof: Consider

$$A_{ij} = \begin{array}{cc} & x \rightarrow \\ \begin{array}{c}\alpha \\ \downarrow\end{array} & \left(\left(\begin{array}{c} y = j \\ \beta = i \end{array} \right) \right) \end{array}$$

$$\therefore A_{ij} = \begin{pmatrix} h(1,1,j,i) & h(2,1,j,i) & \cdots & h(N,1,j,i) \\ h(1,2,j,i) & h(2,2,j,i) & \cdots & h(N,2,j,i) \\ \vdots & \vdots & & \vdots \\ h(1,N,j,i) & h(2,N,j,i) & \cdots & h(N,N,j,i) \end{pmatrix}$$

Shift-invariant $\Leftrightarrow$ $h(x, \alpha, y, \beta) = g(\alpha - x, \beta - y)$ for some g.

$$\therefore A_{ij} = \begin{pmatrix} g(0, i-j) & g(\overset{N-1}{\cancel{-1}}, i-j) & \cdots & g(\overset{1}{\cancel{1-N}}, i-j) \\ h(1, i-j) & g(0, i-j) & \cdots & g(\overset{2}{\cancel{2-N}}, i-j) \\ \vdots & \vdots & & \vdots \\ h(N-1, i-j) & g(N-2, i-j) & \cdots & g(0, i-j) \end{pmatrix} \leftarrow \text{Circulant}$$

(Assume periodic property)

Properties of separable image transformation

Recall: Separable $h \Leftrightarrow h(x,\alpha,y,\beta) = h_c(x,\alpha)\, h_r(y,\beta)$.

Let $g = Hf$.

H: transformation matrix

$$\Longrightarrow g(\alpha,\beta) = \sum_{x=1}^{N} h_c(x,\alpha) \underbrace{\sum_{y=1}^{N} f(x,y)\, h_r(y,\beta)}_{\text{Matrix multiplication}}$$

Consider

$$h_r = (h_r(y,\beta))_{1\le y,\beta\le N} \in M_{N\times N}$$

$$h_c = (h_c(x,\alpha))_{1\le x,\alpha\le N} \in M_{N\times N} \quad \text{Let } S = f\, h_r.$$

$$f = (f(x,y))_{1\le x,y\le N} \in M_{N\times N}$$

Easy to see: $g(\alpha,\beta) = \sum_{x=1}^{N} h_c(x,\alpha)\, S(x,\beta) = \sum_{x=1}^{N} h_c^T(\alpha,x)\, S(x,\beta)$

$\therefore\ g = h_c^T S = h_c^T f\, h_r$ (Matrix form)

Image decomposition

Suppose $h(x,\alpha,y,\beta) = h_c(x,\alpha)\, h_r(y,\beta)$ (Separable).

Then: $g = h_c^T f\, h_r \Rightarrow f = (h_c^T)^{-1} g\, (h_r)^{-1}$

Write: $(h_c^T)^{-1} = \begin{pmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_N \\ | & | & & | \end{pmatrix}$; $h_r^{-1} = \begin{pmatrix} — & \vec{v}_1^T & — \\ — & \vec{v}_2^T & — \\ & \vdots & \\ — & \vec{v}_N^T & — \end{pmatrix}$

Then: $f = \underbrace{\sum_{i=1}^{N} \sum_{j=1}^{N} g_{ij}\, \vec{u}_i \vec{v}_j^T}$ $M_{N\times N}$

Check that: $(h_c^T)^{-1} \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \underset{(i,j)\text{-entry}}{1} & \vdots \\ 0 & \cdots & 0 \end{pmatrix} h_r^{-1} = \vec{u}_i \vec{v}_j^T$

$\therefore f =$ linear combination of $\{\vec{u}_i \vec{v}_j^T\}_{i,j}$

Example: We see that $f = \sum_{i=1}^{N} \sum_{j=1}^{N} g_{ij} \vec{u}_i \vec{v}_j^T$

Assume g is a diagonal matrix (that is, $g_{ij} = 0$ if $i \neq j$) and suppose only r diagonal entries are non-zero.

Then:

$$f = \sum_{i=1}^{r} g_{ii} \vec{u}_i \vec{v}_i^T$$

The storage requirement is: $(\underset{\substack{\uparrow \\ \vec{u}_i}}{N} + \underset{\substack{\uparrow \\ \vec{v}_i}}{N} + \underset{\substack{\uparrow \\ g_{ii}}}{1}) \times r$

Similarity between images

Need to define matrix norm $\|\cdot\|$ such that: for $\forall f, g \in \mathcal{I}$, we can define similarity between f and g as $\|f-g\|$.

Definition: A vector/matrix norm is a function $\|\cdot\| : \mathbb{R}^m$ (or $\mathbb{R}^{m\times n}$) $\to \mathbb{R}$ so that for any $\vec{x}, \vec{y} \in \mathbb{R}^m$ (or $\mathbb{R}^{m\times n}$) and $\alpha \in \mathbb{R}$, we have:

1. $\|\vec{x}\| \geq 0$, $\|\vec{x}\| = 0$ iff $\vec{x} = 0$.
2. $\|\vec{x}+\vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$ (triangle inequality)
3. $\|\alpha \vec{x}\| = |\alpha|\,\|\vec{x}\|$

Example:

$$\left.\begin{aligned} &\bullet\ \|\vec{x}\|_1 = \sum_{i=1}^{m} |x_i| \qquad \vec{x} = (x_1, x_2, \dots, x_m)^T \\ &\bullet\ \|\vec{x}\|_2 = \left(\sum_{i=1}^{m} x_i^2\right)^{1/2} \\ &\bullet\ \|\vec{x}\|_\infty = \max_{i=1,2,\dots,m} |x_i| \end{aligned}\right\} \text{Vector norm}$$