Lecture 1:

Image transformation

Let $\mathcal{I}$ = Collection of images of size N and range of intensity $[0, M]$.

$$= \{ f \in M_{N \times N}(\mathbb{R}) : 0 \leq f(i,j) \leq M ; 1 \leq i, j \leq N \}$$

(for simplicity, assume f is a square image; can be general $N_1 \times N_2$ image)

Image transformation = $\mathcal{O} : \mathcal{I} \to \mathcal{I}$ (transform one image to another)

Imaging problems

① Find a suitable transformation $T \Rightarrow$ $g := T(f)$ becomes good!

noisy image

② Given a noisy/distorted image g and transformation T, find original clean image f.

$$\underset{\text{known}}{g} = \underset{\text{known}}{T}(\underset{\text{unknown}}{f}) + \underset{\text{unknown}}{n}$$

(Inverse problem)

Definition: (Linear image transformation)

$\mathcal{O}: \mathcal{I} \to \mathcal{I}$ is linear $\iff$ $\mathcal{O}(af + g) = a\,\mathcal{O}(f) + \mathcal{O}(g)$ for $\forall f, g \in \mathcal{I}$; $\forall a \in \mathbb{R}$

Take $f \in \mathcal{I}$. Let

$$f = \begin{pmatrix} f(1,1) & \cdots & f(1,N) \\ f(2,1) & & f(2,N) \\ \vdots & f(i,j) & \vdots \\ f(N,1) & \cdots & f(N,N) \end{pmatrix} = \sum_{i=1}^{N} \sum_{j=1}^{N} \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & f(i,j) & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} = \sum_{i=1}^{N} \sum_{j=1}^{N} f(x,y) \begin{array}{cc} \begin{array}{ccccc} & & \downarrow y & & \end{array} & \\ \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} & \leftarrow x \end{array}$$

Let $g = \mathcal{O}(f)$. Assume $\mathcal{O}$ is linear, then:

$$g(\alpha, \beta) = \sum_{x=1}^{N} \sum_{y=1}^{N} f(x,y) \left[\mathcal{O}\left(\left(\begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} \right) \right) \right]_{\alpha, \beta}$$

$$= \sum_{x=1}^{N} \sum_{y=1}^{N} f(x,y)\, h(x, \alpha, y, \beta) \quad \text{where}$$

$$h(x, \alpha, y, \beta) = \left[\mathcal{O}(P_{xy}) \right]_{\alpha, \beta} \; ; \; P_{xy} = \begin{array}{cc} \begin{array}{ccccc} & & y^{th} \downarrow & & \end{array} & \\ \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} & \leftarrow x^{th} \end{array}$$

Remark: $h(x,\alpha,y,\beta)$ = how much input value at (x,y) influence the output value at (α,β).

Pixel (x,y) affecting pixel (α,β) by a weight $h(x,\alpha,y,\beta)$.

Definition: (Point spread function)

$h(\cdot,\alpha,\cdot\,\beta)$ is called the PSF at (α,β).

Fix α,β. Let x,y as variables!

Definition: (Shift-invariant)

Shift invariant $\Longleftrightarrow$ $h(x,\alpha,y,\beta) = h(\alpha-x,\beta-y)$ for $\forall\, 1\leq x,y,\alpha,\beta\leq N$

Definition: (Convolution) Let $f, g \in \mathcal{I}$.

Convolution of f and g $\Leftrightarrow$ $f * g(\alpha, \beta) = \sum_{x=1}^{N} \sum_{y=1}^{N} f(x,y)\, g(\alpha - x, \beta - y)$

(Assume f and g are periodically extended: $\begin{cases} f(x+iN, y+jN) = f(x,y) \\ g(x+iN, y+jN) = g(x,y) \end{cases}$ $\forall\, i,j \in \mathbb{Z}$)

Theorem: PSF is shift-invariant $\Rightarrow$ the operator $\mathcal{O}$ is a convolution with the input image.

Proof: Let $g := \mathcal{O}(f)$. $g(\alpha, \beta) = \sum_{x=1}^{N} \sum_{y=1}^{N} f(x,y)\, \underbrace{h(x, \alpha, y, \beta)}_{h(\alpha - x,\, \beta - y)}$

$= f * h(\alpha, \beta)$

Remark:

- $f * h = h * f$ (exercise)
- Convolution is important for understanding image blur.

Definition: (Separable)

Separable $\Leftrightarrow$ $h(x,\alpha,y,\beta) = h_c(x,\alpha)\, h_r(y,\beta)$ for $\forall$ $1 \le x, y, \alpha, \beta \le N$.

Theorem: Suppose PSF is separable. Then:

The operator $\mathcal{O}$ consists of two matrix multiplication.

Proof:

$$g(\alpha,\beta) = \sum_{x=1}^{N}\sum_{y=1}^{N} f(x,y)\, h(x,\alpha,y,\beta)$$

$$= \sum_{x=1}^{N}\sum_{y=1}^{N} f(x,y)\, h_c(x,\alpha)\, h_r(y,\beta)$$

$$= \underbrace{\sum_{x=1}^{N} h_c(x,\alpha)\underbrace{\left(\sum_{y=1}^{N} f(x,y)\, h_r(y,\beta)\right)}_{\text{matrix multiplication}}}_{\text{matrix multiplication}}$$

$$\boxed{\text{Recall: } (AB)_{ij} = \sum_{k=1}^{N} A_{ik} B_{kj}}$$