MMAT5390: Mathematical Image Processing

Assignment 1

Due: 18 February 2021

Please give reasons in your solutions.

1. Determine if the following PSFs of linear transformations on $N \times N$ square images are separable and/or shift-invariant (with h_s being N-periodic in both arguments) by observing their corresponding transformation matrices:

   (a) $H = \begin{pmatrix} a & b & r & s \\ b & d & t & u \\ r & s & a & c \\ t & u & b & d \end{pmatrix}$, where $a, b, c, d, r, s, t, u > 0$, $b \neq c$ and $ct - bs \neq 0$;

   (b) $H = \begin{pmatrix} 0 & 0 & 0 & 2 & 4 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 & 10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 0 & 0 & 0 & 3 & 6 & 9 \\ 4 & 5 & 0 & 0 & 0 & 0 & 12 & 15 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 & 18 & 0 & 0 \\ 0 & 0 & 0 & 4 & 8 & 12 & 0 & 0 & 0 \\ 0 & 0 & 0 & 16 & 20 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 24 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$;

   (c) $H = \begin{pmatrix} \pi & 2\pi & 3\pi & 4\pi \\ 2\pi & \pi & 4\pi & 3\pi \\ 3\pi & 4\pi & \pi & 2\pi \\ 4\pi & 3\pi & 2\pi & \pi \end{pmatrix}$;

   (d) $H = \begin{pmatrix} 9 & 9 & 18 & 9 & 9 & 18 & 18 & 18 & 36 \\ 18 & 9 & 9 & 18 & 9 & 9 & 36 & 18 & 18 \\ 9 & 18 & 9 & 9 & 18 & 9 & 18 & 36 & 18 \\ 18 & 18 & 36 & 9 & 9 & 18 & 9 & 9 & 18 \\ 36 & 18 & 18 & 18 & 9 & 9 & 18 & 9 & 9 \\ 18 & 36 & 18 & 9 & 18 & 9 & 9 & 18 & 9 \\ 9 & 9 & 18 & 18 & 18 & 36 & 9 & 9 & 18 \\ 18 & 9 & 9 & 36 & 18 & 18 & 18 & 9 & 9 \\ 9 & 18 & 9 & 18 & 36 & 18 & 9 & 18 & 9 \end{pmatrix}$.

2. Suppose A and B are two matrices. The **Kronecker product** $A \otimes B$ is defined as:

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1N}B \\ a_{21}B & \cdots & a_{2N}B \\ \vdots & & \vdots \\ a_{N1}B & \cdots & a_{NN}B \end{pmatrix},$$

   where a_{ij} is the i-th row, j-th column entry of A.

   Consider a linear image transformation, whose PSF is separable and given by: $h(x, \alpha, y, \beta) = h_c(x, \alpha) h_r(y, \beta)$. Show that the transformation matrix H is given by:

$$H = h_r^T \otimes h_c^T.$$

3. Let $f, g \in M_{m \times n}(\mathbb{R})$, and assume that they are periodically extended. Prove that $f * g = g * f$.

4. Let $\mathcal{O}$ be a linear image transformation on $M_{n\times n}(\mathbb{R})$ whose PSF h is shift-invariant with h_s being n-periodic in both arguments. Prove that its corresponding transformation matrix H is block-circulant.

5. (a) Let $A = \begin{pmatrix} 3 & 3 & 6 \\ 0 & 1 & 2 \\ 0 & 1 & 8 \end{pmatrix}$ and let $B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. What is the value of α that minimizes $\|A - \alpha B\|_F$?

   (b) Let $C = \begin{pmatrix} 2 & 3 & 5 & 7 \\ 8 & 6 & 4 & 2 \end{pmatrix}$ and let $D = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$. What is the value of α that minimizes $\|C - \alpha D\|_F$?

   (c) Which central measures (mean, median, mode) of the pixel values are the values of α that respectively minimize the Frobenius norm difference? Please explain your answer.

6. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$.

   (a) Compute an SVD of A. Please show all your steps.

   (b) Write A as a linear combination of its elementary images from SVD. Please show all your steps.

7. Consider a matrix $A \in M_{M\times N}(\mathbb{R})$, and let

$$A = U\Sigma V^T$$

   be one of its singular value decompositions, such that $\sigma_{ii} \geq \sigma_{jj}$ whenever $i < j$.

   (a) Show that the K-tuple $(\sigma_{11}, \sigma_{22}, \ldots, \sigma_{KK})$, where $K = \min\{M, N\}$, is uniquely determined.

   (b) Show that if $\{\sigma_{ii} : i = 1, 2, \ldots, K\}$ are distinct *and nonzero*, then the *first* K columns of U and V are uniquely determined up to a change of sign. In other words, for each $i = 1, 2, \ldots, K$, there are exactly two choices of $(\vec{u}_i, \vec{v}_i)$; denoting one choice by $(\vec{u}, \vec{v})$, the other is given by $(-\vec{u}, -\vec{v})$.

   (c) Does the claim in (b) hold if we drop the assumption? Prove it or give a counterexample.