MATH3360: Mathematical Imaging

Final practice solutions

17 December 2021

Please give detailed steps and reasons in your solutions.

1. (Q3)

   (a) Assuming central spectrum, the Butterworth high-pass filter is

   $$H(u,v) = \frac{1}{1 + (D_0^2/D(u,v))^n}$$

   Similar to HW3 Q1, $H(2,1) = \frac{25}{26}$ and $H(-1,3) = \frac{100}{101}$ gives

   $$D_0 = 1 \text{ and } n = 2$$

   (b) Assuming central spectrum, the Gaussian low-pass filter is

   $$H_G(u,v) = exp\Big(-\frac{D(u,v)}{2\sigma^2}\Big)$$

   So,

   $$H(2,2) = 1 + (1 - exp\Big(-\frac{8}{2\sigma^2}\Big)) = \frac{MN+1}{MN}$$

   Then we have,

   $$\sigma^2 = \frac{4}{\log \frac{MN}{MN-1}}$$

2. (Q4)

   (a)

   $$H_1(u,v) = \sum_{x=-k}^{k}\sum_{y=-k}^{k}\frac{1}{(2k+1)^2}e^{-2\pi j(\frac{ux}{M}+\frac{vy}{N})} = \frac{1}{(2k+1)^2}\sum_{x=-k}^{k}e^{-2\pi j\frac{ux}{M}}\sum_{y=-k}^{k}e^{-2\pi j\frac{vy}{N}}$$
   $$= \frac{1}{(2k+1)^2}[1 + 2\sum_{x=1}^{k}\cos\frac{2\pi ux}{M}][1 + 2\sum_{y=1}^{k}\cos\frac{2\pi vy}{N}].$$

   (b)

   $$\begin{aligned} H_4(u,v) &= -4 + e^{-2\pi j\frac{u}{M}} + e^{2\pi j\frac{u}{M}} + e^{-2\pi j\frac{v}{N}} + e^{2\pi j\frac{v}{N}} \\ &= -4 + 2\cos\frac{2\pi u}{M} + 2\cos\frac{2\pi v}{N} \\ &= -4(\sin^2\frac{\pi u}{M} + \sin^2\frac{\pi v}{N}). \end{aligned}$$

(c)

$$\begin{aligned}
H_5(u,v) &= \frac{1}{T}\sum_{t=0}^{T-1} e^{-2\pi j(\frac{atu}{M}+\frac{btv}{N})} \\
&= \begin{cases} \frac{1}{T}\cdot\frac{1-e^{-2\pi jT(\frac{au}{M}+\frac{bv}{N})}}{1-e^{-2\pi j(\frac{au}{M}+\frac{bv}{N})}} & \text{if } \frac{au}{M}+\frac{bv}{N}\notin\mathbb{Z}, \\ 1 & \text{otherwise,} \end{cases} \\
&= \begin{cases} \frac{1}{T}e^{-\pi j(T-1)(\frac{au}{M}+\frac{bv}{N})}\frac{e^{\pi jT(\frac{au}{M}+\frac{bv}{N})}-e^{-\pi jT(\frac{au}{M}+\frac{bv}{N})}}{e^{\pi j(\frac{au}{M}+\frac{bv}{N})}-e^{-\pi j(\frac{au}{M}+\frac{bv}{N})}} & \text{if } \frac{au}{M}+\frac{bv}{N}\notin\mathbb{Z}, \\ 1 & \text{otherwise,} \end{cases} \\
&= \begin{cases} \frac{1}{T}e^{-\pi j(T-1)(\frac{au}{M}+\frac{bv}{N})}\frac{\sin(\pi T(\frac{au}{M}+\frac{bv}{N}))}{\sin(\pi(\frac{au}{M}+\frac{bv}{N}))} & \text{if } \frac{au}{M}+\frac{bv}{N}\notin\mathbb{Z}, \\ 1 & \text{otherwise.} \end{cases}
\end{aligned}$$

3. (Q5)

$$\begin{aligned}
&\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} F(m,n)\overline{F(m,n)} \\
&= \frac{1}{M^2N^2}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1} f(k,l)e^{-2\pi j(\frac{mk}{M}+\frac{nl}{N})}\sum_{k'=0}^{M-1}\sum_{l'=0}^{N-1}\overline{f(k',l')}e^{2\pi j(\frac{mk'}{M}+\frac{nl'}{N})} \\
&= \frac{1}{M^2N^2}\sum_{m,k,k'=0}^{M-1}\sum_{n,l,l'=0}^{N-1} f(k,l)\overline{f(k',l')}e^{2\pi j(\frac{m(k'-k)}{M}+\frac{n(l'-l)}{N})} \\
&= \frac{1}{M^2N^2}\sum_{k,k'=0}^{M-1}\sum_{l,l'=0}^{N-1} f(k,l)\overline{f(k',l')}\cdot M\mathbf{1}_{M\mathbb{Z}}(k'-k)\cdot N\mathbf{1}_{N\mathbb{Z}}(l'-l) \\
&= \frac{1}{MN}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1}|f(k,l)|^2.
\end{aligned}$$

4. (Q7)

(a) For any $f\in M_{N\times N}(\mathbb{R})$ and $x,y\in\{0,1,\cdots,N-1\}$,

$$\begin{aligned}
h_1 * f(x,y) &= \sum_{k=0}^{N-1}\sum_{l=0}^{N-1} h_1(k,l)f(x-k,y-l) \\
&= f(x+1,y)-f(x,y).
\end{aligned}$$

Then for any $f\in M_{N\times N}(\mathbb{R})$ and $x,y\in\{0,1,\cdots,N-1\}$,

$$\begin{aligned}
&\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} h_1 * h_1(k,l)f(x-k,y-l) \\
&= (h_1 * h_1) * f(x,y) \\
&= h_1 * (h_1 * f)(x,y) \\
&= h_1 * f(x+1,y) - h_1 * f(x,y) \\
&= f(x+2,y)-f(x+1,y)-[f(x+1,y)-f(x,y)] \\
&= f(x+2,y)-2f(x+1,y)+f(x,y).
\end{aligned}$$

Hence $h_1 * h_1(x,y) = \begin{cases} 1 & \text{if } (x,y) = (0,0) \text{ or } (-2,0), \\ -2 & \text{if } (x,y) = (-1,0), \\ 0 & \text{otherwise.} \end{cases}$

(b) For any $f \in M_{N\times N}(\mathbb{R})$ and $x, y \in \{0, 1 \cdots, N-1\}$,

$$\begin{aligned} h_2 * f(x,y) &= \sum_{k=0}^{N-1}\sum_{l=0}^{N-1} h_2(k,l) f(x-k, y-l) \\ &= \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f(x-k, y-l). \end{aligned}$$

Then for any $f \in M_{N\times N}(\mathbb{R})$ and $x, y \in \{0, 1 \cdots, N-1\}$,

$$\begin{aligned} &\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} h_2 * h_2(k,l) f(x-k, y-l) \\ &= (h_2 * h_2) * f(x,y) \\ &= h_2 * (h_2 * f)(x,y) \\ &= \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} h_2 * f(x-k, y-l) \\ &= \frac{1}{81}\sum_{k,k',l,l'=-1}^{1} f(x-k-k', y-l-l'). \end{aligned}$$

Note that for $k, k' \in \{-1, 0, 1\}$,
$k + k' = -2$ if and only if $(k, k') = (-1, -1)$,
$k + k' = -1$ if and only if $(k, k') = (-1, 0)$ or $(0, -1)$,
$k + k' = 0$ if and only if $(k, k') = (-1, 1)$ or $(0, 0)$ or $(1, -1)$,
$k + k' = 1$ if and only if $(k, k') = (0, 1)$ or $(1, 0)$,
$k + k' = 2$ if and only if $(k, k') = (1, 1)$.
The same goes for $l + l'$. Hence

$$h_2 * h_2(x,y) = \begin{cases} \frac{1}{81}(3-|x|)(3-|y|) & \text{if } x, y \in \{-2,-1,0,1,2\}, \\ 0 & \text{otherwise.} \end{cases}$$

(c) For any $f \in M_{N\times N}(\mathbb{R})$ and $x, y \in \{0, 1, \cdots, N-1\}$,

$$\begin{aligned} h_3 * f(x,y) &= \sum_{k=0}^{N-1}\sum_{l=0}^{N-1} h_3(k,l) f(x-k, y-l) \\ &= \frac{1}{2} f(x,y) + \frac{1}{4}[f(x+1,y) + f(x-1,y) + f(x,y+1) + f(x,y-1)]. \end{aligned}$$

Then for any $f \in M_{N\times N}(\mathbb{R})$ and $x, y \in \{0, 1, \cdots, N-1\}$,

$$
\begin{aligned}
&\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} h_3 * h_3(k,l) f(x-k, y-l)\\
&= (h_3 * h_3) * f(x,y)\\
&= h_3 * (h_3 * f)(x,y)\\
&= \frac{1}{2} h_3 * f(x,y) + \frac{1}{4}[h_3 * f(x+1,y) + h_3 * f(x-1,y) + h_3 * f(x,y+1) + h_3 * f(x,y-1)]\\
&= \frac{1}{2}\{\frac{1}{2} f(x,y) + \frac{1}{4}[f(x+1,y) + f(x-1,y) + f(x,y+1) + f(x,y-1)]\}\\
&+ \frac{1}{4}\{\frac{1}{2} f(x+1,y) + \frac{1}{4}[f(x+2,y) + f(x,y) + f(x+1,y+1) + f(x+1,y-1)]\}\\
&+ \frac{1}{4}\{\frac{1}{2} f(x-1,y) + \frac{1}{4}[f(x,y) + f(x-2,y) + f(x-1,y+1) + f(x-1,y-1)]\}\\
&+ \frac{1}{4}\{\frac{1}{2} f(x,y+1) + \frac{1}{4}[f(x+1,y+1) + f(x-1,y+1) + f(x,y+2) + f(x,y)]\}\\
&+ \frac{1}{4}\{\frac{1}{2} f(x,y-1) + \frac{1}{4}[f(x+1,y-1) + f(x-1,y-1) + f(x,y) + f(x,y-2)]\}\\
&= \frac{1}{2} f(x,y) + \frac{1}{4}[f(x+1,y) + f(x-1,y) + f(x,y+1) + f(x,y-1)]\\
&\quad + \frac{1}{8}[f(x+1,y+1) + f(x-1,y+1) + f(x+1,y-1) + f(x-1,y-1)]\\
&\quad + \frac{1}{16}[f(x+2,y) + f(x-2,y) + f(x,y+2) + f(x,y-2)].
\end{aligned}
$$

Hence

$$
h_3 * h_3(x,y) = \begin{cases} \frac{1}{2} & \text{if } D(x,y) = 0,\\ \frac{1}{4} & \text{if } D(x,y) = 1,\\ \frac{1}{8} & \text{if } D(x,y) = 2,\\ \frac{1}{16} & \text{if } D(x,y) = 4,\\ 0 & \text{otherwise.} \end{cases}
$$

(d) One crucial point in this course is

Convolution in the spatial domain $\Longleftrightarrow$ Direct product in the frequency domain

Also,

Direct product in the spatial domain $\Longleftrightarrow$ Convolution in the frequency domain

Since we have two equivalent ways to compute the same result, in practice we can always choose the one that make computation more efficient, with the help of FFT or etc.
In this question, we can simply take

$$H_i = N^2 DFT(h_i)$$

5. (a) $E(f) = \int_\Omega |f(x,y) - g(x,y)|^2 + \|\nabla f(x,y)\|^4 \, dx\, dy$. Then for any $\varphi : \Omega \to \mathbb{R}$,

$$
\begin{aligned}
\frac{\partial E(f+t\varphi)}{\partial t}\Big|_{t=0} &= \frac{\partial}{\partial t}\Big|_{t=0} \int_\Omega \left\{ |f(x,y) + t\varphi(x,y) - g(x,y)|^2 + K(x,y)\|\nabla(f+t\varphi)(x,y)\|^4 \right\} dx\, dy\\
&= \int_\Omega 2\varphi(f-g) + 4K\|\nabla f\|^2 \nabla f \cdot \nabla\varphi \; dx\, dy\\
&= \int_\Omega 2\varphi(f-g)\, dx\, dy - \int_\Omega 4\varphi\nabla\cdot(K\|\nabla f\|^2 \nabla f)\, dx\, dy + \int_\Omega 4\varphi K \|\nabla f\|^2 \nabla f \cdot \vec{n}\, d\sigma
\end{aligned}
$$

So, the minimizer of the energy must satisfy the following PDEs

$$\begin{cases} f(x,y) - g(x,y) - 2\nabla \cdot (K\|\nabla f\|^2 \nabla f)(x,y) = 0 & \text{if } (x,y) \in \Omega \\ \big\|\nabla f(x,y)\big\|^2 \nabla f(x,y) \cdot \vec{n}(x,y) = 0 & \text{if } (x,y) \in \partial\Omega \end{cases}$$

Ignoring the boundary condition, we then have the following iterative scheme in the continuous case

$$f^{n+1}(x,y) = f^n(x,y) + t\Big(- f^n + g + 2\nabla \cdot (K\|\nabla f^n\|^2 \nabla f^n)\Big)(x,y)$$

for small $t > 0$. In the discrete case, we can adopt the following finite difference approximation

$$\begin{aligned} \nabla \cdot (K\|\nabla f\|^2 \nabla f)(u,v) \approx & \left(K\|\nabla f\|^2 \frac{\partial f}{\partial x}\right)(u+1,v) - \left(K\|\nabla f\|^2 \frac{\partial f}{\partial x}\right)(u,v) \\ & + \left(K\|\nabla f\|^2 \frac{\partial f}{\partial y}\right)(u,v+1) - \left(K\|\nabla f\|^2 \frac{\partial f}{\partial y}\right)(u,v) \end{aligned}$$

$$\frac{\partial f}{\partial x}(u,v) \approx f(u+1,v) - f(u,v)$$
$$\frac{\partial f}{\partial y}(u,v) \approx f(u,v+1) - f(u,v)$$

(b) $E(f) = \displaystyle\int_\Omega \sum_{i=1}^N (f(x,y) - g_i(x,y))^2 + \lambda\|\nabla f(x,y)\|^2 \, dx\, dy$. Then for any $\varphi : \Omega \to \mathbb{R}$,

$$\begin{aligned} \frac{\partial E(f+t\varphi)}{\partial t}\bigg|_{t=0} &= \frac{\partial}{\partial t}\bigg|_{t=0} \int_\Omega \sum_{i=1}^N (f + t\varphi - g_i)^2 + \lambda\|\nabla f + t\nabla\varphi\|^2 \, dx\, dy \\ &= \int_\Omega 2\sum_{i=1}^N \varphi(f - g_i) + 2\lambda\nabla f \cdot \nabla\varphi \, dx\, dy \\ &= \int_\Omega 2\sum_{i=1}^N \varphi(f - g_i) - 2\lambda\varphi\,\Delta f \, dx\, dy + \int_{\partial\Omega} 2\lambda\varphi\,\nabla f \cdot \vec{n}\, d\sigma \end{aligned}$$

So, the minimizer of the energy must satisfy the following PDEs

$$\begin{cases} \sum_{i=1}^N \big(f(x,y) - g_i(x,y)\big) - \lambda\Delta f(x,y) = 0 & \text{if } (x,y) \in \Omega \\ \nabla f(x,y) \cdot \vec{n}(x,y) = 0 & \text{if } (x,y) \in \partial\Omega \end{cases}$$

Ignoring the boundary condition, we then have the following iterative scheme in the continuous case

$$f^{n+1}(x,y) = f^n(x,y) + t\left(\sum_{i=1}^N \big(- f(x,y) + g_i(x,y)\big) + \lambda\Delta f(x,y)\right)$$

In the discrete case, we can adopt the following finite difference approximation

$$\Delta f(u,v) \approx f(u+1,v) + f(u-1,v) + f(u,v+1) + f(u,v-1) - 4f(u,v)$$

(c) (Class notes 19 and lecture notes of chapter 4 are good references for this question.)
$E(f) = \int_\Omega (h * f(x,y) - g(x,y))^2 + K(x,y)\big\|\nabla f(x,y)\big\|\; dx\, dy$. Then for any $\varphi : \Omega \to \mathbb{R}$,

$$\begin{aligned}\frac{\partial E(f+t\varphi)}{\partial t}\Big|_{t=0} &= \frac{\partial}{\partial t}\Big|_{t=0} \int_\Omega (h*f + t\, h*\varphi - g)^2 + K\|\nabla f + t\nabla\varphi\|\; dx\, dy \\ &= \int_\Omega 2(h*f-g)\, h*\varphi + K\frac{1}{\|\nabla f\|}\nabla f\cdot\nabla\varphi\, dx\, dy \\ &= \int_\Omega 2\tilde{H}(h*f-g)\,\varphi - \varphi\nabla\cdot\Big(K\frac{1}{\|\nabla f\|}\nabla f\Big)\, dx\, dy + \int_{\partial\Omega}\varphi K\frac{1}{\|\nabla f\|}\nabla f\cdot\vec{n}\, d\sigma\end{aligned}$$

where $\tilde{H}$ is an operator defined by

$$\tilde{H}(\phi)(x,y) = \int_\Omega h(\alpha,\beta)\phi(\alpha+x,\beta+y)\, d\alpha\, d\beta$$

So, the minimizer of the energy must satisfy the following PDEs

$$\begin{cases} 2\tilde{H}(h*f-g)(x,y) - \nabla\cdot\Big(K\frac{1}{\|\nabla f\|}\nabla f\Big)(x,y) = 0 & \text{if } (x,y)\in\Omega \\ K(x,y)\frac{1}{\big\|\nabla f(x,y)\big\|}\nabla f(x,y)\cdot\vec{n}(x,y) = 0 & \text{if } (x,y)\in\partial\Omega\end{cases}$$

Ignoring the boundary condition, we then have the following iterative scheme in the continuous case

$$f^{n+1}(x,y) = f^n(x,y) + t\Big(\nabla\cdot\Big(K\frac{1}{\|\nabla f^n\|}\nabla f^n\Big)(x,y) - 2\tilde{H}(h*f^n - g)(x,y)\Big)$$

The numerical approximations can be done exactly the same as the two examples above.

6. (Q9)

(a) Let $\gamma \in C^4_{2\pi}([0,2\pi],\Omega)$ with $\gamma(s) = (\gamma_1(s),\gamma_2(s))^T$ be a minimizer of E. Then for any $g \in C^2_{2\pi}([0,2\pi],\Omega)$,

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}\Big|_{t=0} E(\gamma+tg) = \frac{\partial}{\partial t}\Big|_{t=0}\int_0^{2\pi}\Big[\|(\gamma+tg)'(s)\|^2 + \lambda\|(\gamma+tg)''(s)\|^2 + \mu V((\gamma+tg)(s))\Big]\, ds \\ &= \int_0^{2\pi}\frac{\partial}{\partial t}\Big|_{t=0}\Big[\|\gamma'(s)\|^2 + 2t\langle\gamma'(s),g'(s)\rangle + t^2\|g'(s)\|^2 + \lambda\Big(\|\gamma''(s)\|^2 + 2t\langle\gamma''(s),g''(s)\rangle + t^2\|\gamma''(s)\|^2\Big) \\ &\quad + \mu V((\gamma+tg)(s))\Big]\, ds \\ &= \int_0^{2\pi}\big[2\langle\gamma'(s),g'(s)\rangle + 2\lambda\langle\gamma''(s),g''(s)\rangle + \langle\mu\nabla V(\gamma(s)),g(s)\rangle\big]\, ds \\ &= \int_0^{2\pi}\langle 2\lambda\gamma^{(4)}(s) - 2\gamma''(s) + \mu\nabla V(\gamma(s)), g(s)\rangle\, ds \\ &\quad + [2\langle\gamma'(s),g(s)\rangle + 2\lambda(\langle\gamma''(s),g'(s)\rangle - \langle\gamma^{(3)},g(s)\rangle)]\Big|_{s=0}^{2\pi} \\ &= \int_{\substack{s\in[0,2\pi]\\ \|\gamma(s)\|\geq 1}}\langle 2\lambda\gamma^{(4)}(s) - 2\gamma''(s) + 2\mu\gamma(s), g(s)\rangle\, ds \\ &\quad + \int_{\substack{s\in[0,2\pi]\\ \|\gamma(s)\|<1}}\langle 2\lambda\gamma^{(4)}(s) - 2\gamma''(s) - \mu\Big(\frac{2\gamma_1(s)}{([\gamma_1(s)]^2+\varepsilon)^2}, \frac{2\gamma_2(s)}{([\gamma_2(s)]^2+\varepsilon)^2}\Big)^T, g(s)\rangle\, ds, \quad (***)\end{aligned}$$

which implies that γ satisfies:

$$\begin{aligned} 0 &= 2\lambda\gamma^{(4)}(s) - 2\gamma''(s) + \mu\nabla V(\gamma(s)) \\ &= \begin{cases} 2\lambda g^{(4)}(s) - 2\gamma''(s) + 2\mu\gamma(s) & \text{if } \|\gamma(s)\| \geq 1, \\ 2\lambda g^{(4)}(s) - 2\gamma''(s) - \mu(\frac{2\gamma_1(s)}{([\gamma_1(s)]^2+\varepsilon)^2}, \frac{2\gamma_2(s)}{([\gamma_2(s)]^2+\varepsilon)^2})^T & \text{if } \|\gamma(s)\| < 1. \end{cases} \end{aligned}$$

(b) Hence $(***)$ gives a descent direction:

$$g(s) = \begin{cases} -\lambda\gamma^{(4)}(s) + \gamma''(s) - \mu\gamma(s) & \text{if } \|\gamma(s)\| \geq 1 \\ -\lambda\gamma^{(4)}(s) + \gamma''(s) + \mu(\frac{\gamma_1(s)}{([\gamma_1(s)]^2+\varepsilon)^2}, \frac{\gamma_2(s)}{([\gamma_2(s)]^2+\varepsilon)^2})^T & \text{if } \|\gamma(s)\| < 1, \end{cases}$$

which gives rise to the following gradient descent scheme:

$$\frac{\partial\gamma(s;t)}{\partial t} = \begin{cases} -\lambda\frac{\partial^4\gamma}{\partial s^4}(s;t) + \frac{\partial^2\gamma(s;t)}{\partial s^2} - \mu\gamma(s;t) & \text{if } \|\gamma(s;t)\| \geq 1 \\ -\lambda\frac{\partial^4\gamma}{\partial s^4}(s;t) + \frac{\partial^2\gamma}{\partial s^2}(s;t) + \mu(\frac{\gamma_1(s;t)}{([\gamma_1(s;t)]^2+\varepsilon)^2}, \frac{\gamma_2(s;t)}{([\gamma_2(s;t)]^2+\varepsilon)^2})^T & \text{if } \|\gamma(s;t)\| < 1. \end{cases}$$

with the following (explicit Euler) discretization for a closed contour of N points:

$$\begin{aligned} &\frac{\gamma^{n+1}(s) - \gamma^n(s)}{\Delta t} \\ &= \begin{cases} -\lambda\frac{\gamma^n(s+2)-4\gamma^n(s+1)+6\gamma^n(s)-4\gamma^n(s-1)+\gamma^n(s-2)}{\sigma^4} \\ \quad + \frac{\gamma^n(s+1)-2\gamma^n(s)+\gamma^n(s-1)}{\sigma^2} - 2\mu\gamma^n(s) & \text{if } \|\gamma^n(s)\| \geq 1, \\ \\ -\lambda\frac{\gamma^n(s+2)-4\gamma^n(s+1)+6\gamma^n(s)-4\gamma^n(s-1)+\gamma^n(s-2)}{\sigma^4} \\ \quad + \frac{\gamma^n(s+1)-2\gamma^n(s)+\gamma^n(s-1)}{\sigma^2} + \mu(\frac{\gamma_1^n(s)}{([\gamma_1^n(s)]^2+\varepsilon)^2}, \frac{\gamma_2^n(s)}{([\gamma_2^n(s)]^2+\varepsilon)^2})^T & \text{if } \|\gamma^n(s)\| < 1. \end{cases} \end{aligned}$$

Letting $D' = \begin{pmatrix} 6 & -4 & 1 & 0 & \cdots & 0 & 1 & -4 \\ -4 & 6 & -4 & 1 & \cdots & 0 & 0 & 1 \\ 1 & -4 & 6 & -4 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -4 & 6 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & -4 & 6 & -4 \\ -4 & 1 & 0 & 0 & \cdots & 1 & -4 & 6 \end{pmatrix} \in M_{N\times N}(\mathbb{R})$ and D as in the previous example,

$$\gamma^{n+1} = \begin{cases} [(1-2\mu\Delta t)I - \frac{\lambda\Delta t}{\sigma^4}D' + \frac{\Delta t}{\sigma^2}D]\gamma^n & \text{if } \|\gamma^n(s)\| \geq 1\,\forall s, \\ [I - \frac{\lambda\Delta t}{\sigma^4}D' + \frac{\Delta t}{\sigma^2}D]\gamma^n + \mu\Delta t(\frac{\gamma_1^n(s)}{([\gamma_1^n(s)]^2+\varepsilon)^2}, \frac{\gamma_2^n(s)}{([\gamma_1^n(s)]^2+\varepsilon)^2})^T & \text{if } \|\gamma^n(s)\| < 1\,\forall s. \end{cases}$$

Since D' is also circulant, its eigenvalues are known to be:

$$\begin{aligned} \lambda_i' &= \sum_{k=0}^{N-1} d_k' \exp(2\pi j\frac{ki}{N}) = 6 - 8\cos\frac{2\pi i}{N} + 2\cos\frac{4\pi i}{N} \\ &= 4(\cos\frac{2\pi i}{N} - 1)^2 \in [0, 16]. \end{aligned}$$

Hence given initial contour γ^0 enclosing the unit circle, if we choose $0 < \Delta t < \frac{\sigma^2(\sigma^2+1)}{2\mu\sigma^4+4\sigma^2+16\lambda}$, then the contour will be attracted towards the unit circle.

Remark: You may try to find the range of $(\lambda, \mu, \Delta t)$ such that the portion of the contour that lies inside the unit circle would be attracted to expand to the unit circle. They are also encouraged to discretize the functional E in the latter example, and investigate whether the correspondingly derived gradient descent scheme coincides with the above.