# MATH3360: Mathematical Imaging

## Assignment 1

Due: 1 Oct 2021

Please give reasons in your solutions.

1. Determine if the following PSFs of linear transformations on $N \times N$ square images are separable and/or shift-invariant (with $h_s$ being $N$-periodic in both arguments) by observing their corresponding transformation matrices:

   (a) $H = \begin{pmatrix} 2 & 0 & 2 & 2 \\ 4 & 2 & 0 & 4 \\ 4 & 2 & 0 & 4 \\ 2 & 0 & 2 & 2 \end{pmatrix}$

   (b) $H = \begin{pmatrix} 9 & 9 & 18 & 9 & 9 & 18 & 18 & 18 & 36 \\ 18 & 9 & 9 & 18 & 9 & 9 & 36 & 18 & 18 \\ 9 & 18 & 9 & 9 & 18 & 9 & 18 & 36 & 18 \\ 18 & 18 & 36 & 9 & 9 & 18 & 9 & 9 & 18 \\ 36 & 18 & 18 & 18 & 9 & 9 & 18 & 9 & 9 \\ 18 & 36 & 18 & 9 & 18 & 9 & 9 & 18 & 9 \\ 9 & 9 & 18 & 18 & 18 & 36 & 9 & 9 & 18 \\ 18 & 9 & 9 & 36 & 18 & 18 & 18 & 9 & 9 \\ 9 & 18 & 9 & 18 & 36 & 18 & 9 & 18 & 9 \end{pmatrix}.$

   (c) $H(x, \alpha, y, \beta) = (\alpha - x)(\beta - y) + (\alpha - x)^2$

   (d) $H(x, \alpha, y, \beta) = \alpha\beta e^{(x-y)(x^2+xy+y^2)}$

2. Let $f, g \in M_{m\times n}(\mathbb{R})$, and assume that they are periodically extended. Prove that $f * g = g * f$.

3. Let $\mathcal{O}$ be a linear image transformation on $M_{n\times n}(\mathbb{R})$ whose PSF $h$ is shift-invariant. Prove that its corresponding transformation matrix $H$ is block Toeplitz.
   *Remark*: a Toeplitz matrix or diagonal-constant matrix, is a matrix in which each descending diagonal from left to right is constant. For example, the following matrix $T$ is Toeplitz

$$T = \begin{pmatrix} a & b & c & d \\ e & a & b & c \\ f & e & a & b \\ g & f & e & a \end{pmatrix}$$

A block Toeplitz matrix is another special kind of block matrix, which contains blocks that are repeated down the diagonals of the matrix, as a Toeplitz matrix has elements repeated down the diagonal. The individual block matrix elements, $A_{ij}$, must also be a Toeplitz matrix. A block toeplitz matrix has the form

$A = \begin{pmatrix} A_0 & A_{-1} & \cdots & \\ A_1 & A_0 & A_{-1} & \cdots \\ A_2 & A_1 & A_0 & \cdots \\ \cdots & \cdots & & \end{pmatrix}$, where matrices $A_0, A_1, \cdots$ are Toeplitz.

Search on google or Wikipedia if you want to know more details about block Toeplitz matrices.

4. Consider a linear image transformation on $M_{n\times n}(\mathbb{R})$, whose PSF is separable and given by: $h(x, \alpha, y, \beta) = h_c(x, \alpha)h_r(y, \beta)$. Show that the transformation matrix $H$ is given by:

$$H = h_r^T \otimes h_c^T.$$

5. This question is about the singular value decomposition: consider

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

(a) Write $A$ as a linear combination of eigen-images.
(b) Find an image $\tilde{A}$ with rank 2 such that $\|\tilde{A} - A\|_F$ is equal to 2 ($\|\cdot\|_F$ is the Frobenius norm). Please explain your answer with details.

6. **Coding assignment**: Please read the 4 MATLAB files in the attached zip file carefully. There are several lines missing in these 4 files. Add the missing lines by yourself and test the files using the given image. (Note: In this coding assignment, we discuss the image processing of grayscale images only.)

**Coding instruction**:

**Q1**: In this question, you are required to complete the code to construct the $N^2 \times N^2$ matrix $H$. This matrix is so sparse (most of the elements are zero) that directly constructing such a matrix will consume a large amount of memory. We would like to use a MATLAB built-in function called *sparse* to implement the algorithm.

The usage of *sparse* is as follow:

$$H = sparse(i, j, v, m, n)$$

which generates an $m \times n$ sparse matrix $H$ from the triplets i, j, and v such that $H(i(k), j(k)) = v(k)$.

For example, if we want to construct a $4 \times 4$ matrix:

$$H = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 6 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

The code looks like

$$H = sparse(rows, cols, vals, 4, 4)$$

where $rows = [1, 2, 3, 4]$, $cols = [3, 4, 2, 1]$ and $vals = [2, 9, 6, 1]$.

The relation is as follows:

$$H(1,3) = 2 \tag{1}$$
$$H(2,4) = 9 \tag{2}$$
$$H(3,2) = 6 \tag{3}$$
$$H(4,1) = 1 \tag{4}$$

In the code, we can combine the three variables to a $4 \times 3$ array,

$$ijv = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 4 & 9 \\ 3 & 2 & 6 \\ 4 & 1 & 1 \end{pmatrix}$$

where the three columns are the variable $rows$, $cols$ and $vals$ respectively.

You should find out the values and their corresponding indices in the $H$ matrix and constructing the array $ijv$. Given a $512 \times 512$ image and $3 \times 3$ kernel, the shape of $ijv$ should be $(512^2 \times 3 \times 3, 3)$, because each row has $3 \times 3$ elements coming from the kernel, and there is $512^2$ rows. Then use $ijv$ and the built-in function $sparse$ to create the matrix $H$.

**Q2**: Theorem: If a PSF is separable and shift-invariant, then the transform can be decomposed into two 1D convolutions.

For example, given a $5 \times 5$ image $I$ and a kernel $k$

$$I = \begin{pmatrix} 9 & 8 & 2 & 0 & 1 \\ 7 & 5 & 3 & 9 & 2 \\ 0 & 6 & 7 & 2 & 3 \\ 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix} \qquad k = \begin{pmatrix} \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \end{pmatrix}$$

Under the periodically extended assumption, the convolution $I * k$ can also be written in the separable transform.

$$I * k = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 9 & 8 & 2 & 0 & 1 \\ 7 & 5 & 3 & 9 & 2 \\ 0 & 6 & 7 & 2 & 3 \\ 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

**Q3**: Use the $U$, $S$ and $V$ matrix obtained from *svd* function to generate rank-40 approximation, where $S$ is the diagonal matrix and the columns of $U$ and $V$ form two orthonormal bases.

**Q4**: The inverse Haar transform can be written as follows

$$I(m,n) = \sum_{i=1}^{N} \tilde{H}^T(m,i) \sum_{j=1}^{N} \hat{I}(i,j)\tilde{H}(j,n) \tag{5}$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} \hat{I}(i,j)\tilde{H}^T(m,i)\tilde{H}(j,n) \tag{6}$$

where $I$ and $\hat{I}$ are the reconstructed image and the Haar coefficients respectively. From the above equations, we know that the transform can be rewritten as the summation of weighted elementary images.