MATH3360: Mathematical Imaging
Solutions to Chapter 3 Exercises

1. $H_i = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$, $H_{ii} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$,

$H_{iii} = \begin{pmatrix} 1 & \frac{16}{17} & \frac{1}{2} & \frac{16}{17} \\ \frac{16}{17} & \frac{4}{5} & \frac{16}{41} & \frac{4}{5} \\ \frac{1}{2} & \frac{16}{41} & \frac{1}{5} & \frac{16}{41} \\ \frac{16}{17} & \frac{4}{5} & \frac{16}{41} & \frac{4}{5} \end{pmatrix}$, $H_{iv} = \begin{pmatrix} 0 & \frac{1}{17} & \frac{1}{2} & \frac{1}{17} \\ \frac{1}{17} & \frac{5}{25} & \frac{4}{41} & \frac{5}{25} \\ \frac{1}{2} & \frac{4}{41} & \frac{5}{5} & \frac{4}{41} \\ \frac{1}{17} & \frac{1}{5} & \frac{25}{41} & \frac{1}{5} \end{pmatrix}$,

$H_v = \begin{pmatrix} 1 & e^{-\frac{1}{8}} & e^{-\frac{1}{2}} & e^{-\frac{1}{8}} \\ e^{-\frac{1}{8}} & e^{-\frac{1}{4}} & e^{-\frac{5}{8}} & e^{-\frac{1}{4}} \\ e^{-\frac{1}{2}} & e^{-\frac{5}{8}} & e^{-1} & e^{-\frac{5}{8}} \\ e^{-\frac{1}{8}} & e^{-\frac{1}{4}} & e^{-\frac{5}{8}} & e^{-\frac{1}{4}} \end{pmatrix}$, $H_{vi} = \begin{pmatrix} 0 & 1-e^{-\frac{1}{8}} & 1-e^{-\frac{1}{2}} & 1-e^{-\frac{1}{8}} \\ 1-e^{-\frac{1}{8}} & 1-e^{-\frac{1}{4}} & 1-e^{-\frac{5}{8}} & 1-e^{-\frac{1}{4}} \\ 1-e^{-\frac{1}{2}} & 1-e^{-\frac{5}{8}} & 1-e^{-1} & 1-e^{-\frac{5}{8}} \\ 1-e^{-\frac{1}{8}} & 1-e^{-\frac{1}{4}} & 1-e^{-\frac{5}{8}} & 1-e^{-\frac{1}{4}} \end{pmatrix}$.

(a)

$$DFT(f_1) = \frac{1}{16} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix}$$

$$= \frac{1}{16} \begin{pmatrix} 4 & 4 & 4 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

i,iii,v.
$$\tilde{f}_1 = Re(iDFT(DFT(f_1))) = f_1.$$

ii,iv,vi.
$$\tilde{f}_1 = Re(iDFT(\mathbf{0})) = \mathbf{0}.$$

(b)

$$DFT(f_2) = \frac{1}{16} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix}$$

$$= \frac{1}{16} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & -4 & 4 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

i.
$$\tilde{f}_2 = Re(iDFT(\mathbf{0})) = \mathbf{0}.$$

ii.
$$\tilde{f}_2 = Re(iDFT(DFT(f_2))) = f_2.$$

iii.
$$\tilde{f}_2 = Re(iDFT(\frac{1}{5}DFT(f_2))) = \frac{1}{5}f_2.$$

iv.
$$\tilde{f}_2 = Re(iDFT(\frac{4}{5}DFT(f_2))) = \frac{4}{5}f_2.$$

v.
$$\tilde{f}_2 = Re(iDFT(e^{-1}DFT(f_2))) = e^{-1}f_2.$$

vi.
$$\tilde{f}_2 = Re(iDFT((1 - e^{-1})DFT(f_2))) = (1 - e^{-1})f_2.$$

(c)

$$DFT(f_3) = \frac{1}{16}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix}$$

$$= \frac{1}{16}\begin{pmatrix} 0 & 2 & 2 & 0 \\ 0 & -1-j & -1-j & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1+j & -1+j & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix}$$

$$= \frac{1}{16}\begin{pmatrix} 4 & -2-2j & 0 & -2+2j \\ -2-2j & 2j & 0 & 2 \\ 0 & 0 & 0 & 0 \\ -2+2j & 2 & 0 & -2j \end{pmatrix}.$$

i.
$$\tilde{f}_3 = iDFT(DFT(f_3)) = f_3.$$

ii.
$$\tilde{f}_3 = iDFT(\mathbf{0}) = \mathbf{0}.$$

iii.

$$\tilde{f}_3 = \frac{1}{16} Re(iDFT\left(\begin{pmatrix} 4 & -\frac{32}{17}(1+j) & 0 & \frac{32}{17}(-1+j) \\ -\frac{32}{17}(1+j) & \frac{8}{5}j & 0 & \frac{8}{5} \\ 0 & 0 & 0 & 0 \\ \frac{32}{17}(-1+j) & \frac{8}{5} & 0 & -\frac{8}{5}j \end{pmatrix}\right))$$

$$= \frac{1}{16} Re(\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{pmatrix}\begin{pmatrix} 4 & -\frac{32}{17}(1+j) & 0 & \frac{32}{17}(-1+j) \\ -\frac{32}{17}(1+j) & \frac{8}{5}j & 0 & \frac{8}{5} \\ 0 & 0 & 0 & 0 \\ \frac{32}{17}(-1+j) & \frac{8}{5} & 0 & -\frac{8}{5}j \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & \\ 1 & j & -1 & - \\ 1 & -1 & 1 & \\ 1 & -j & -1 & \end{pmatrix}$$

$$= \frac{1}{340}\begin{pmatrix} -7 & 17 & 17 & -7 \\ 17 & 313 & 313 & 17 \\ 17 & 313 & 313 & 17 \\ -7 & 17 & 17 & -7 \end{pmatrix}.$$

iv.

$$\tilde{f}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \frac{1}{340}\begin{pmatrix} -7 & 17 & 17 & -7 \\ 17 & 313 & 313 & 17 \\ 17 & 313 & 313 & 17 \\ -7 & 17 & 17 & -7 \end{pmatrix}$$

$$= \frac{1}{340}\begin{pmatrix} 7 & -17 & -17 & 7 \\ -17 & 27 & 27 & -17 \\ -17 & 27 & 27 & -17 \\ 7 & -17 & -17 & 7 \end{pmatrix}.$$

v.

$$\tilde{f}_3 = \frac{1}{16} Re(iDFT\left(\begin{pmatrix} 4 & -2e^{-\frac{1}{8}}(1+j) & 0 & 2e^{-\frac{1}{8}}(-1+j) \\ -2e^{-\frac{1}{8}}(1+j) & 2e^{-\frac{1}{4}}j & 0 & 2e^{-\frac{1}{4}} \\ 0 & 0 & 0 & 0 \\ 2e^{-\frac{1}{8}}(-1+j) & 2e^{-\frac{1}{4}} & 0 & -2e^{-\frac{1}{4}}j \end{pmatrix}\right))$$

$$\approx \begin{pmatrix} 0.0035 & 0.0553 & 0.0553 & 0.0035 \\ 0.0553 & 0.8859 & 0.8859 & 0.0553 \\ 0.0553 & 0.8859 & 0.8859 & 0.0553 \\ 0.0035 & 0.0553 & 0.0553 & 0.0035 \end{pmatrix}.$$

vi.

$$\tilde{f}_3 \approx \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0.0035 & 0.0553 & 0.0553 & 0.0035 \\ 0.0553 & 0.8859 & 0.8859 & 0.0553 \\ 0.0553 & 0.8859 & 0.8859 & 0.0553 \\ 0.0035 & 0.0553 & 0.0553 & 0.0035 \end{pmatrix}$$

$$= \begin{pmatrix} -0.0035 & -0.0553 & -0.0553 & -0.0035 \\ -0.0553 & 0.1141 & 0.1141 & -0.0553 \\ -0.0553 & 0.1141 & 0.1141 & -0.0553 \\ -0.0035 & -0.0553 & -0.0553 & -0.0035 \end{pmatrix}$$

3

2. (a) Since $M \geq 6$ and $N \geq 8$, $D(1,2) = 5$, $D(3,1) = 10$ and $D(2,4) = 20$.
Since $DFT(f)(1,2) \neq 0$, $DFT(f)(3,1) \neq 0$ and $DFT(f)(2,4) \neq 0$,

$$3H_1(1,2) = 4H_1(3,1) = 6H_1(2,4), \text{ i.e. } \frac{3D_0^{2n}}{D_0^{2n} + 5^n} = \frac{4D_0^{2n}}{D_0^{2n} + 10^n} = \frac{6D_0^{2n}}{D_0^{2n} + 20^n}.$$

Since $D_0 > 0$,

$$\begin{cases} 3(D_0^{2n} + 10^n) &= 4(D_0^{2n} + 5^n), \\ 4(D_0^{2n} + 20^n) &= 6(D_0^{2n} + 10^n), \end{cases}$$

which gives

$$\begin{cases} D_0^{2n} &= 3 \cdot 10^n - 4 \cdot 5^n = 5^n(3 \cdot 2^n - 4), \\ 2D_0^{2n} &= 4 \cdot 20^n - 6 \cdot 10^n \implies D_0^{2n} = 2 \cdot 20^n - 3 \cdot 10^n = 5^n(2 \cdot 2^{2n} - 3 \cdot 2^n). \end{cases}$$

Hence $3 \cdot 2^n - 4 = 2 \cdot 2^{2n} - 3 \cdot 2^n$, and thus $(2^n - 1)(2^n - 2) = 0$ and $n = 0$ or 1.
If $n = 0$, then $1 = D_0^{2 \times 0} = 5^0(3 \cdot 2^0 - 4) = -1$. Contradiction.
Hence $n = 1$. Since $D_0^2 = 5(3 \cdot 2 - 4)$, $D_0 = \sqrt{10}$.

(b) Since $M \geq 2c > 2a$ and $N \geq 4c$, $D(a,b) = a^2 + b^2 = c^2$ and $D(0, 2c) = 4c^2$.
Given $H_2(a,b) = 8H_2(0, 2c)$, i.e. $e^{-\frac{c^2}{2\sigma^2}} = 8e^{-\frac{2c^2}{\sigma^2}}$.
Hence $e^{-\frac{c^2}{2\sigma^2}}(2e^{-\frac{c^2}{2\sigma^2}} - 1)(4e^{-\frac{c^2}{\sigma^2}} + 2e^{-\frac{c^2}{2\sigma^2}} + 1) = 0$.
Thus $e^{-\frac{c^2}{2\sigma^2}} = \frac{1}{2}$; $-\frac{c^2}{2\sigma^2} = -\ln 2$ and thus $\sigma = \frac{c}{\sqrt{2\ln 2}}$.

(c) Since $M = N = 10$, $D(2,7) = 2^2 + 3^2 = 13$ and $D(5,1) = 5^2 + 1^2 = 26$.
Given $H_3(2,7) = \frac{1}{3}$ and $H_3(5,1) = \frac{4}{5}$, i.e.

$$\begin{cases} \frac{13^n}{D_0^{2n} + 13^n} &= \frac{1}{3}, \\ \frac{26^n}{D_0^{2n} + 26^n} &= \frac{4}{5}. \end{cases}$$

Then $D_0^{2n} = 2 \cdot 13^n$ and $4D_0^{2n} = 26^n$, and thus $4 = 2^{n-1}$.
Hence $n = 3$; then $D_0^6 = D_0^{2n} = 2 \cdot 13^n = 2 \cdot 13^3$, and thus $D_0 = \sqrt[6]{2\sqrt{13}}$.

3. Recall that for any $f \in M_{M \times N}(\mathbb{R})$, $DFT(h*f)(u,v) = MN\,DFT(h)(u,v)DFT(f)(u,v)$; hence $H(u,v) = MN\,DFT(h)(u,v)$.

(a)

$$H_1(u,v) = \sum_{x=-k}^{k} \sum_{y=-k}^{k} \frac{1}{(2k+1)^2} e^{-2\pi j(\frac{ux}{M} + \frac{vy}{N})} = \frac{1}{(2k+1)^2} \sum_{x=-k}^{k} e^{-2\pi j \frac{ux}{M}} \sum_{y=-k}^{k} e^{-2\pi j \frac{vy}{N}}$$

$$= \frac{1}{(2k+1)^2}[1 + 2\sum_{x=1}^{k} \cos \frac{2\pi ux}{M}][1 + 2\sum_{y=1}^{k} \cos \frac{2\pi vy}{N}].$$

4

(b) $H_2(u,v) = \frac{r}{r+4} + \frac{1}{r+4}(e^{-2\pi j \frac{u}{M}} + e^{2\pi j \frac{u}{M}} + e^{-2\pi j \frac{v}{N}} + e^{2\pi j \frac{v}{N}}) = \frac{r + 2(\cos \frac{2\pi u}{M} + \cos \frac{2\pi v}{N})}{r+4}.$

(c)

$$
\begin{aligned}
H_3(u,v) &= \frac{1}{4} + \frac{1}{8}(e^{-2\pi j \frac{u}{M}} + e^{2\pi j \frac{u}{M}} + e^{-2\pi j \frac{v}{N}} + e^{2\pi j \frac{v}{N}}) \\
&\quad + \frac{1}{16}(e^{-2\pi j(\frac{u}{M} + \frac{v}{N})} + e^{-2\pi j(\frac{u}{M} - \frac{v}{N})} + e^{-2\pi j(-\frac{u}{M} + \frac{v}{N})} + e^{-2\pi j(-\frac{u}{M} - \frac{v}{N})}) \\
&= \frac{1}{4} + \frac{1}{4}(\cos \frac{2\pi u}{M} + \cos \frac{2\pi v}{N}) + \frac{1}{4}\cos \frac{2\pi u}{M} \cos \frac{2\pi v}{N} \\
&= \frac{1}{4}(\cos \frac{2\pi u}{M} + 1)(\cos \frac{2\pi v}{N} + 1) \\
&= \cos^2 \frac{\pi u}{M} \cos^2 \frac{\pi v}{N}.
\end{aligned}
$$

(d)

$$
\begin{aligned}
H_4(u,v) &= -4 + e^{-2\pi j \frac{u}{M}} + e^{2\pi j \frac{u}{M}} + e^{-2\pi j \frac{v}{N}} + e^{2\pi j \frac{v}{N}} \\
&= -4 + 2\cos \frac{2\pi u}{M} + 2\cos \frac{2\pi v}{N} \\
&= -4(\sin^2 \frac{\pi u}{M} + \sin^2 \frac{\pi v}{N}).
\end{aligned}
$$

(e)

$$
\begin{aligned}
H_5(u,v) &= \frac{1}{T}\sum_{t=0}^{T-1} e^{-2\pi j(\frac{atu}{M} + \frac{btv}{N})} \\
&= \begin{cases} \frac{1}{T} \cdot \frac{1 - e^{-2\pi jT(\frac{au}{M} + \frac{bv}{N})}}{1 - e^{-2\pi j(\frac{au}{M} + \frac{bv}{N})}} & \text{if } \frac{au}{M} + \frac{bv}{N} \notin \mathbb{Z}, \\ 1 & \text{otherwise}, \end{cases} \\
&= \begin{cases} \frac{1}{T} e^{-\pi j(T-1)(\frac{au}{M} + \frac{bv}{N})} \frac{e^{\pi jT(\frac{au}{M} + \frac{bv}{N})} - e^{-\pi jT(\frac{au}{M} + \frac{bv}{N})}}{e^{\pi j(\frac{au}{M} + \frac{bv}{N})} - e^{-\pi j(\frac{au}{M} + \frac{bv}{N})}} & \text{if } \frac{au}{M} + \frac{bv}{N} \notin \mathbb{Z}, \\ 1 & \text{otherwise}, \end{cases} \\
&= \begin{cases} \frac{1}{T} e^{-\pi j(T-1)(\frac{au}{M} + \frac{bv}{N})} \frac{\sin(\pi T(\frac{au}{M} + \frac{bv}{N}))}{\sin(\pi(\frac{au}{M} + \frac{bv}{N}))} & \text{if } \frac{au}{M} + \frac{bv}{N} \notin \mathbb{Z}, \\ 1 & \text{otherwise}. \end{cases}
\end{aligned}
$$

4. (a)

$$
\begin{aligned}
DFT(g_1) &= \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 5 & 5 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} 10 & 0 \\ 0 & 2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}. \\
DFT(h_1) &= \frac{1}{16}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{16}\begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\
&= \frac{1}{16}\begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned}
$$

i.

$$DFT(\tilde{f}_1)(u,v) = \frac{DFT(g_1)(u,v)}{MN[DFT(h_1)(u,v) + \varepsilon \mathrm{sgn}(DFT(h_1)(u,v))]}$$

and thus $DFT(\tilde{f}_1) = \dfrac{1}{2}\begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix} \odot \dfrac{1}{2 \times 2}\begin{pmatrix} (\frac{1}{4} + \varepsilon)^{-1} & \frac{1}{\varepsilon} \\ \frac{1}{\varepsilon} & \frac{1}{\varepsilon} \end{pmatrix}$

$$= \frac{1}{8}\begin{pmatrix} \frac{20}{1+4\varepsilon} & 0 \\ 0 & \frac{1}{\varepsilon} \end{pmatrix}.$$

Hence

$$\tilde{f}_1 = \frac{1}{2}iDFT(\begin{pmatrix} \frac{5}{1+\varepsilon} & 0 \\ 0 & \frac{1}{\varepsilon} \end{pmatrix})$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \frac{5}{1+\varepsilon} & 0 \\ 0 & \frac{1}{\varepsilon} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} \frac{5}{1+\varepsilon} & \frac{1}{\varepsilon} \\ \frac{5}{1+\varepsilon} & -\frac{1}{\varepsilon} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \frac{5}{1+\varepsilon} + \frac{1}{\varepsilon} & \frac{5}{1+\varepsilon} - \frac{1}{\varepsilon} \\ \frac{5}{1+\varepsilon} - \frac{1}{\varepsilon} & \frac{5}{1+\varepsilon} + \frac{1}{\varepsilon} \end{pmatrix}.$$

ii. $B = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{5} \end{pmatrix}$; hence

$$DFT(\tilde{f}_1)(u,v) = \frac{B(u,v)DFT(g_1)(u,v)}{MN[DFT(h_1)(u,v) + \varepsilon \mathrm{sgn}(DFT(h)(u,v))]}$$

and thus $DFT(\tilde{f}_1) = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{5} \end{pmatrix} \odot \dfrac{1}{2}\begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix} \odot \dfrac{1}{2 \times 2} \times 4\begin{pmatrix} \frac{1}{1+\varepsilon} & \frac{1}{\varepsilon} \\ \frac{1}{\varepsilon} & \frac{1}{\varepsilon} \end{pmatrix}$

$$=$$

(b)   i.

ii.

iii.

(c)

5.

$$\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}F(m,n)\overline{F(m,n)}$$

$$=\frac{1}{M^2N^2}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1}f(k,l)e^{-2\pi j(\frac{mk}{M}+\frac{nl}{N})}\sum_{k'=0}^{M-1}\sum_{l'=0}^{N-1}\overline{f(k',l')}e^{2\pi j(\frac{mk'}{M}+\frac{nl'}{N})}$$

$$=\frac{1}{M^2N^2}\sum_{m,k,k'=0}^{M-1}\sum_{n,l,l'=0}^{N-1}f(k,l)\overline{f(k',l')}e^{2\pi j(\frac{m(k'-k)}{M}+\frac{n(l'-l)}{N})}$$

$$=\frac{1}{M^2N^2}\sum_{k,k'=0}^{M-1}\sum_{l,l'=0}^{N-1}f(k,l)\overline{f(k',l')}\cdot M\mathbf{1}_{M\mathbb{Z}}(k'-k)\cdot N\mathbf{1}_{N\mathbb{Z}}(l'-l)$$

$$=\frac{1}{MN}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1}|f(k,l)|^2.$$

6. (a) Let f be a minimizer of $\|L\vec{f}\|_F^2$ over all $\{\varphi\in M_{M\times N}(\mathbb{R}):\|\vec{g}-D\vec{f}\|_2^2=\varepsilon\}$. Then the method of Lagrange multipliers states that f necessarily minimizes the following Lagrangian:

$$\mathcal{L}(f)=\|L\vec{f}\|_2^2+\lambda\|\vec{g}-D\vec{f}\|_2^2.$$

Hence $\frac{d\mathcal{L}}{d\vec{f}}=0$, i.e.

$$\frac{d}{d\vec{f}}[\vec{f}^TL^TL\vec{f}+\lambda(\vec{g}^T-\vec{f}^TD^T)(\vec{g}-D\vec{f})]=0,$$

which gives $2L^TL\vec{f}+\lambda(-2D^T\vec{g}+2D^TD\vec{f})=0$ and thus $(\lambda D^TD+L^TL)\vec{f}=\lambda D^T\vec{g}$.

(b) i. For any $x,y\in\{0,1,\cdots,M-1\}$,

$$(\sqrt{M}U_M)\overline{(\sqrt{M}U_M)}(x,y)=M\sum_{k=0}^{M-1}U_M(x,k)\overline{U_M(k,y)}=\frac{1}{M}\sum_{k=0}^{M-1}e^{2\pi j\frac{k(y-x)}{M}}$$

$$=\frac{1}{M}\cdot M\delta(y-x)=\begin{cases}1 & \text{if }x=y,\\0 & \text{otherwise.}\end{cases}$$

Hence $\sqrt{M}U_M$ is unitary.

ii. Note that the (k,l)-th block of $U\otimes V$, denoted here by $(U\otimes V)_{k,l}$, is given by $U(k,l)V$. Hence the (k,l)-th block of $(U\otimes V)^*$ is given by

$$[(U\otimes V)^*]_{k,l}=[(U\otimes V)_{l,k}]^*=\overline{U(l,k)}V^*.$$

Hence for any $k, l \in \{0, 1, \cdots, N\}$,

$$[(U \otimes V)(U \otimes V)^*]_{k,l} = \sum_{n=0}^{N-1} (U \otimes V)_{k,n}[(U \otimes V)^*]_{n,l}$$

$$= \sum_{n=0}^{N-1} U(k,n)\overline{U(n,l)}VV^*$$

$$= \delta(k-l)I_M.$$

Hence $U \otimes V$ is unitary.

iii. For any $k, l \in \{0, 1, \cdots, M\}$,

$$U_M C \overline{U_M}(k,l) = \sum_{m=0}^{M-1} U_M(k,m)(C\overline{U_M})(m,l)$$

$$= \sum_{m=0}^{M-1}\sum_{n=0}^{M-1} U_M(k,m)c_{m-n}\overline{U_M}(n,l)$$

$$= \frac{1}{M^2} \sum_{m=0}^{M-1}\sum_{n=0}^{M-1} c_{m-n}e^{2\pi j\frac{ln-mk}{M}}$$

$$= \frac{1}{M^2} \sum_{m=0}^{M-1}\sum_{n'=m-M+1}^{m} c_{n'}e^{2\pi j\frac{l(m-n')-mk}{M}}$$

$$= \frac{1}{M^2} \sum_{n'=0}^{M-1} c_{n'}e^{-2\pi j\frac{ln'}{M}} \sum_{m=0}^{M-1} e^{2\pi j\frac{m(l-k)}{M}}$$

$$= \frac{1}{M} \sum_{n=0}^{M-1} c_n e^{-2\pi j\frac{ln}{M}}\delta(l-k).$$

Hence $U_M C\overline{U_M}$ is diagonal.

Since $\sqrt{M}U_M$ is unitary, $U_M^{-1} = MU_M^* = M\overline{U_M}$, and thus $MU_M C\overline{U_M}$ is a diagonalization of C. Hence the eigenvalues of C are given by the diagonal entries of $MU_M C\overline{U_M}$:

$$\lambda_l(C) = \sum_{n=0}^{M-1} c_n e^{-2\pi j\frac{ln}{M}}, \ l = 0, 1, \cdots, M-1.$$

iv. For any $k, l \in \{0, 1, \cdots, N-1\}$, the (k,l)-th block of $(U_N \otimes U_M)D(\overline{U_N} \otimes \overline{U_M})$, denoted here by $[(U_N \otimes U_M)D(\overline{U_N} \otimes \overline{U_M})]_{k,l}$,

8

is given by:

$$[(U_N \otimes U_M)D(\overline{U_N} \otimes \overline{U_M})]_{k,l} = \sum_{m=0}^{N-1} (U_N \otimes U_M)_{k,m}[D(\overline{U_N} \otimes \overline{U_M})]_{m,l}$$

$$= \sum_{m=0}^{N-1} (U_N \otimes U_M)_{k,m} \sum_{n=0}^{N-1} D_{m-n}(\overline{U_N} \otimes \overline{U_M})_{n,l}$$

$$= \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} U_N(k,m)\overline{U_N(n,l)}U_M D_{m-n}\overline{U_M}$$

$$= \frac{1}{N^2} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} e^{2\pi j \frac{ln-km}{N}} U_M D_{m-n}\overline{U_M}$$

$$= \frac{1}{N^2} \sum_{m=0}^{N-1} \sum_{n'=m-N+1}^{m} e^{2\pi j \frac{l(m-n')-km}{N}} U_M D_{n'}\overline{U_M}$$

$$= \frac{1}{N^2} \sum_{m=0}^{N-1} e^{2\pi j \frac{m(l-k)}{N}} \sum_{n'=0}^{N-1} e^{-2\pi j \frac{ln'}{N}} U_M D_{n'}\overline{U_M}$$

$$= \frac{1}{N}\delta(l-k) \sum_{n'=0}^{N-1} e^{-2\pi j \frac{ln'}{N}} U_M D_{n'}\overline{U_M}.$$

Since D is block-circulant, $D_{n'}$ is circulant. Hence by the result of (b)iii., $U_M D_{n'}\overline{U_M}$ is diagonal and thus $(U_N \otimes U_M)D(\overline{U_N}\otimes\overline{U_M})$ is diagonal.

Since $\sqrt{N}U_N$ and $\sqrt{M}U_M$ are unitary, $\sqrt{MN}U_N \otimes U_M$ is unitary, and $MN(U_N \otimes U_M)D(\overline{U_N} \otimes \overline{U_M})$ is a diagonalization of D. The eigenvalues of D are the diagonal entries of $MN(U_N \otimes U_M)D(\overline{U_N} \otimes \overline{U_M})$, i.e.

$$\lambda_i(D) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} D_{n,m} e^{-2\pi j (\frac{km}{M} + \frac{ln}{N})}, \ k \in \{0, 1, \cdots, M-1\}, l \in \{0, 1, \cdots, N-1\},$$

where $D_{n,m}$ denotes the value of the entries on the diagonal of D_n with indices $\{(x,y) : x - y \in m + M\mathbb{Z}\}$.

(c)   i. For any $x, y \in \{0, 1, \cdots, M-1\}$,

$$W_M D_{k-l} \overline{W_M}(x,y) = \sum_{s=0}^{M-1} W_M(x,s)[D_{k-l}\overline{W_M}](s,y)$$

$$= \sum_{s=0}^{M-1} W_M(x,s) \sum_{t=0}^{M-1} D_{k-l}(s,t)\overline{W_M}(t,y)$$

$$= \frac{1}{M} \sum_{s=0}^{M-1} \sum_{t=0}^{M-1} D_{k-l,s-t} e^{2\pi j \frac{sx-ty}{M}}$$

$$= \frac{1}{M} \sum_{s=0}^{M-1} \sum_{t'=s-M+1}^{s} D_{k-l,t'} e^{2\pi j \frac{sx-(s-t')y}{M}}$$

$$= \frac{1}{M} \sum_{s=0}^{M-1} e^{2\pi j \frac{s(x-y)}{M}} \sum_{t'=0}^{M-1} D_{k-l,t'} e^{2\pi j \frac{t'y}{M}}$$

$$= \begin{cases} \sum_{m=0}^{M-1} D_{k-l,m} e^{2\pi j \frac{my}{M}} & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

ii. For any $k, l \in \mathbb{Z} \cap [0, N-1]$,

$$[(W_N \otimes W_M)D(\overline{W_N} \otimes \overline{W_M})]_{k,l} = \sum_{m=0}^{N-1} (W_N \otimes W_M)_{k,m}[D(\overline{W_N} \otimes \overline{W_M})]_{m,l}$$

$$= \sum_{m=0}^{N-1} (W_N \otimes W_M)_{k,m} \sum_{n=0}^{N-1} D_{m-n}(\overline{W_N} \otimes \overline{W_M})_{n,l}$$

$$= \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} W_N(k,m)\overline{W_N}(n,l)W_M D_{m-n}\overline{W_M}$$

$$= \frac{1}{N} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} e^{2\pi j \frac{km-ln}{N}} W_M D_{m-n}\overline{W_M}$$

$$= \frac{1}{N} \sum_{m=0}^{N-1} \sum_{n'=0}^{N-1} e^{2\pi j \frac{km-l(m-n')}{N}} W_M D_{n'}\overline{W_M}$$

$$= \frac{1}{N} \sum_{m=0}^{N-1} e^{2\pi j \frac{m(k-l)}{N}} \sum_{n'=0}^{N-1} e^{2\pi j \frac{ln'}{N}} W_M D_{n'}\overline{W_M}$$

$$= \delta(k-l) \sum_{n=0}^{N-1} e^{2\pi j \frac{ln}{N}} W_M D_n \overline{W_M}.$$

Hence for any $k, l \in \{0, 1, \cdots, N-1\}$ and $x, y \in \{0, 1, \cdots, M-1\}$,

$$[(W_N \otimes W_M)D(\overline{W_N} \otimes \overline{W_M})](x + kM, y + lM)$$
$$= [(W_N \otimes W_M)D(\overline{W_N} \otimes \overline{W_M})]_{k,l}(x,y)$$
$$= \begin{cases} \displaystyle\sum_{n=0}^{N-1} e^{2\pi j \frac{ln}{N}}[W_M D_n \overline{W_M}](x,y) & \text{if } k = l, \\ 0 & \text{otherwise,} \end{cases}$$
$$= \begin{cases} \displaystyle\sum_{n=0}^{N-1} e^{2\pi j \frac{ln}{N}} \sum_{m=0}^{M-1} D_{n,m} e^{2\pi j \frac{mx}{M}} & \text{if } k = l \text{ and } x = y, \\ 0 & \text{otherwise,} \end{cases}$$
$$= \begin{cases} \displaystyle\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} D(m + nM, 0)e^{2\pi j(\frac{mx}{M} + \frac{ln}{N})} & \text{if } k = l \text{ and } x = y, \\ 0 & \text{otherwise,} \end{cases}$$
$$= \begin{cases} \displaystyle\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} h(m, n)e^{2\pi j(\frac{mx}{M} + \frac{kn}{N})} & \text{if } k = l \text{ and } x = y, \\ 0 & \text{otherwise,} \end{cases}$$
$$= \begin{cases} MN\, \overline{DFT(h)(x, k)} & \text{if } k = l \text{ and } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

(d) For any $x \in \{0, 1, \cdots, M-1\}$ and $y \in \{0, 1, \cdots, N-1\}$, and $f \in M_{M \times N}(\mathbb{R})$,

$$[(W_N \otimes W_M)\mathcal{S}(f)](x + yM) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1}(W_N \otimes W_M)(x + yM, m + nM)\mathcal{S}(f)(m + nM)$$
$$= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} W_N(y, n)W_M(x, m)f(m, n)$$
$$= \frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m, n)e^{2\pi j(\frac{mx}{M} + \frac{ny}{N})}$$
$$= \sqrt{MN}\left[\frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m, n)\overline{e^{-2\pi j(\frac{mx}{M} + \frac{ny}{N})}}\right]$$
$$= \sqrt{MN}\,\overline{DFT(f)(x, y)}$$
$$= \sqrt{MN}\,\overline{\mathcal{S}(DFT(f))(x + yM)}.$$

Hence $(W_N \otimes W_M)\mathcal{S}(f) = \sqrt{MN}\mathcal{S}(\overline{DFT(f)})$.

(e) Recall the result from (a):

$$(\lambda D^T D + L^T L)\vec{f} = \lambda D^T \vec{g}.$$

Since D and L are block-circulant, the result from (c)ii. asserts that

$$[(W_N \otimes W_M)D(\overline{W_N} \otimes \overline{W_M})](x+kM, y+lM) = \begin{cases} MN\overline{DFT(h)(x,k)} & \text{if } k = l \text{ and } x = y, \\ 0 & \text{otherwise} \end{cases};$$

equivalently, denoting the $MN \times MN$ diagonal matrix by Λ_D,

$$D = (\overline{W_N} \otimes \overline{W_M})\Lambda_D(W_N \otimes W_M).$$

Similarly,

$$L = (\overline{W_N} \otimes \overline{W_M})\Lambda_L(W_N \otimes W_M).$$

Hence

$$\begin{aligned}
(\lambda D^T D + L^T L)\vec{f} &= (\lambda D^* D + L^* L)\vec{f} \\
&= \{\lambda[(\overline{W_N} \otimes \overline{W_M})\Lambda_D(W_N \otimes W_M)]^*(\overline{W_N} \otimes \overline{W_M})\Lambda_D(W_N \otimes W_M) \\
&\quad + [(\overline{W_N} \otimes \overline{W_M})\Lambda_L(W_N \otimes W_M)]^*(\overline{W_N} \otimes \overline{W_M})\Lambda_L(W_N \otimes W_M)\}\vec{f} \\
&= [\lambda(W_N \otimes W_M)^*\Lambda_D^*\Lambda_D(W_N \otimes W_M) + (W_N \otimes W_M)^*\Lambda_L^*\Lambda_L(W_N \otimes W_M)]\vec{f} \\
&= (\overline{W_N} \otimes \overline{W_M})(\lambda\Lambda_D^*\Lambda_D + \Lambda_L^*\Lambda_L)(W_N \otimes W_M)\vec{f} \\
&= (\overline{W_N} \otimes \overline{W_M})(\lambda\Lambda_D^*\Lambda_D + \Lambda_L^*\Lambda_L)\sqrt{MN}\mathcal{S}(\overline{DFT(f)});
\end{aligned}$$

on the other hand,

$$\begin{aligned}
\lambda D^T \vec{g} &= \lambda D^* \vec{g} \\
&= \lambda(W_N \otimes W_M)^*\Lambda_D^*(\overline{W}_N \otimes \overline{W_M})^*\vec{g} \\
&= \lambda(\overline{W_N} \otimes \overline{W_M})\Lambda_D^*(W_N \otimes W_M)\vec{g} \\
&= \lambda(\overline{W_N} \otimes \overline{W_M})\Lambda_D^*\sqrt{MN}\,\mathcal{S}(\overline{DFT(g)}).
\end{aligned}$$

Hence $(\lambda\Lambda_D^*\Lambda_D + \Lambda_L^*\Lambda_L)\mathcal{S}(\overline{DFT(f)}) = \lambda\Lambda_D^*\mathcal{S}(\overline{DFT(g)})$. By comparing each pair of entries,

$$\begin{aligned}
&(\lambda M^2 N^2 |DFT(h)(u,v)|^2 + M^2 N^2 |DFT(p)(u,v)|^2)\overline{DFT(f)(u,v)} \\
&= \lambda MN DFT(h)(u,v)\overline{DFT(g)(u,v)},
\end{aligned}$$

which yields

$$DFT(f)(u,v) = \frac{\lambda\overline{DFT(h)(u,v)}DFT(g)(u,v)}{MN(\lambda|DFT(h)(u,v)|^2 + |DFT(p)(u,v)|^2)}.$$

7. To be supplemented.

8. The ideal LPF, Butterworth LPF and Gaussian LPF chosen are respectively

$$H_1 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 1 & \frac{16}{17} & \frac{1}{2} & \frac{16}{17} \\ \frac{16}{17} & \frac{4}{5} & \frac{16}{41} & \frac{4}{5} \\ \frac{1}{2} & \frac{16}{41} & \frac{1}{5} & \frac{16}{41} \\ \frac{16}{17} & \frac{4}{5} & \frac{16}{41} & \frac{4}{5} \end{pmatrix} \text{ and } H_3 = \begin{pmatrix} 1 & e^{-\frac{1}{8}} & e^{-\frac{1}{2}} & e^{-\frac{1}{8}} \\ e^{-\frac{1}{8}} & e^{-\frac{1}{4}} & e^{-\frac{5}{8}} & e^{-\frac{1}{4}} \\ e^{-\frac{1}{2}} & e^{-\frac{5}{8}} & e^{-1} & e^{-\frac{5}{8}} \\ e^{-\frac{1}{8}} & e^{-\frac{1}{4}} & e^{-\frac{5}{8}} & e^{-\frac{1}{4}} \end{pmatrix}.$$

(a)

$$
DFT(f_1) = \frac{1}{16}
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & -j & -1 & j \\
1 & -1 & 1 & -1 \\
1 & j & -1 & -j
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & -j & -1 & j \\
1 & -1 & 1 & -1 \\
1 & j & -1 & -j
\end{pmatrix}
$$

$$
= \frac{1}{16}
\begin{pmatrix}
2 & 2 & 2 & 2 \\
-1-j & 1+j & 1+j & -1-j \\
0 & 0 & 0 & 0 \\
-1+j & 1-j & 1-j & -1+j
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & -j & -1 & j \\
1 & -1 & 1 & -1 \\
1 & j & -1 & -j
\end{pmatrix}
$$

(b)

(c)