MATH3360: Mathematical Imaging

Solutions to Chapter 4 Exercises

1. Recall that the result g_{i} of the 3×3 mean filter applied on $f \in M_{M\times N}(\mathbb{R})$ is given by:

$$g_{\text{i}}(x,y) = \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f(x+k, y+l);$$

the result g_{ii} of the 3×3 median filter applied on $f \in M_{M\times N}(\mathbb{R})$ is given by:

$$g_{\text{ii}}(x,y) = \text{median}\{f(x+k,y+l) : k,l \in \{-1,0,1\}\};$$

the result g_{iii} of the convolution filter $\frac{1}{8}\begin{pmatrix} 4 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ applied on $f \in M_{4\times 4}(\mathbb{R})$ is given by:

$$g_{\text{iii}}(x,y) = \frac{1}{2}f(x,y) + \frac{1}{8}[f(x+1,y)+f(x-1,y)+f(x,y+1)+f(x,y-1)].$$

Hence the results from applying the filters on the given images are:

(a) i. $\frac{1}{9}\begin{pmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{pmatrix}$

ii. $\mathbf{0}$

iii. $\frac{1}{4}\begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix}$

(b) i. $\frac{1}{9}\begin{pmatrix} 7 & 6 & 7 & 7 \\ 7 & 7 & 7 & 6 \\ 6 & 7 & 7 & 7 \\ 7 & 7 & 6 & 7 \end{pmatrix}$

ii. $\mathbf{1}$

iii. $\frac{1}{8}\begin{pmatrix} 7 & 7 & 4 & 7 \\ 4 & 7 & 7 & 7 \\ 7 & 7 & 7 & 4 \\ 7 & 4 & 7 & 7 \end{pmatrix}$

(c) i. $\frac{1}{9}\begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}$

ii. $\begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}$

iii. $\mathbf{0}$

2. (a) For any $f \in M_{N\times N}(\mathbb{R})$ and $x, y \in \{0, 1, \cdots, N-1\}$,

$$\begin{aligned} h_1 * f(x,y) &= \sum_{k=0}^{N-1}\sum_{l=0}^{N-1} h_1(k,l) f(x-k, y-l) \\ &= f(x+1,y) - f(x,y). \end{aligned}$$

Then for any $f \in M_{N\times N}(\mathbb{R})$ and $x, y \in \{0, 1, \cdots, N-1\}$,

$$\begin{aligned} &\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} h_1 * h_1(k,l) f(x-k, y-l) \\ &= (h_1 * h_1) * f(x,y) \\ &= h_1 * (h_1 * f)(x,y) \\ &= h_1 * f(x+1,y) - h_1 * f(x,y) \\ &= f(x+2,y) - f(x+1,y) - [f(x+1,y) - f(x,y)] \\ &= f(x+2,y) - 2f(x+1,y) + f(x,y). \end{aligned}$$

Hence $h_1 * h_1(x,y) = \begin{cases} 1 & \text{if } (x,y) = (0,0) \text{ or } (-2,0), \\ -2 & \text{if } (x,y) = (-1,0), \\ 0 & \text{otherwise.} \end{cases}$

(b) For any $f \in M_{N\times N}(\mathbb{R})$ and $x, y \in \{0, 1 \cdots, N-1\}$,

$$\begin{aligned} h_2 * f(x,y) &= \sum_{k=0}^{N-1}\sum_{l=0}^{N-1} h_2(k,l) f(x-k, y-l) \\ &= \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f(x-k, y-l). \end{aligned}$$

Then for any $f \in M_{N\times N}(\mathbb{R})$ and $x, y \in \{0, 1 \cdots, N-1\}$,

$$\begin{aligned}
&\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} h_2 * h_2(k,l) f(x-k, y-l) \\
&= (h_2 * h_2) * f(x,y) \\
&= h_2 * (h_2 * f)(x,y) \\
&= \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} h_2 * f(x-k, y-l) \\
&= \frac{1}{81}\sum_{k,k',l,l'=-1}^{1} f(x-k-k', y-l-l').
\end{aligned}$$

Note that for $k, k' \in \{-1, 0, 1\}$,
$k + k' = -2$ if and only if $(k, k') = (-1, -1)$,
$k + k' = -1$ if and only if $(k, k') = (-1, 0)$ or $(0, -1)$,
$k + k' = 0$ if and only if $(k, k') = (-1, 1)$ or $(0, 0)$ or $(1, -1)$,
$k + k' = 1$ if and only if $(k, k') = (0, 1)$ or $(1, 0)$,
$k + k' = 2$ if and only if $(k, k') = (1, 1)$.
The same goes for $l + l'$. Hence

$$h_2 * h_2(x,y) = \begin{cases} \frac{1}{81}(3-|x|)(3-|y|) & \text{if } x, y \in \{-2,-1,0,1,2\}, \\ 0 & \text{otherwise.} \end{cases}$$

(c) For any $f \in M_{N\times N}(\mathbb{R})$ and $x, y \in \{0, 1, \cdots, N-1\}$,

$$\begin{aligned}
h_3 * f(x,y) &= \sum_{k=0}^{N-1}\sum_{l=0}^{N-1} h_3(k,l) f(x-k, y-l) \\
&= \frac{1}{2} f(x,y) + \frac{1}{4}[f(x+1,y) + f(x-1,y) + f(x,y+1) + f(x,y-1)].
\end{aligned}$$

Then for any $f \in M_{N\times N}(\mathbb{R})$ and $x, y \in \{0, 1, \cdots, N-1\}$,

$$
\begin{aligned}
&\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} h_3 * h_3(k,l) f(x-k, y-l)\\
&= (h_3 * h_3) * f(x,y)\\
&= h_3 * (h_3 * f)(x,y)\\
&= \frac{1}{2} h_3 * f(x,y) + \frac{1}{4}[h_3 * f(x+1,y) + h_3 * f(x-1,y) + h_3 * f(x,y+1) + h_3 * f(x,y-1)]\\
&= \frac{1}{2}\{\frac{1}{2} f(x,y) + \frac{1}{4}[f(x+1,y) + f(x-1,y) + f(x,y+1) + f(x,y-1)]\}\\
&+ \frac{1}{4}\{\frac{1}{2} f(x+1,y) + \frac{1}{4}[f(x+2,y) + f(x,y) + f(x+1,y+1) + f(x+1,y-1)]\}\\
&+ \frac{1}{4}\{\frac{1}{2} f(x-1,y) + \frac{1}{4}[f(x,y) + f(x-2,y) + f(x-1,y+1) + f(x-1,y-1)]\}\\
&+ \frac{1}{4}\{\frac{1}{2} f(x,y+1) + \frac{1}{4}[f(x+1,y+1) + f(x-1,y+1) + f(x,y+2) + f(x,y)]\}\\
&+ \frac{1}{4}\{\frac{1}{2} f(x,y-1) + \frac{1}{4}[f(x+1,y-1) + f(x-1,y-1) + f(x,y) + f(x,y-2)]\}\\
&= \frac{1}{2} f(x,y) + \frac{1}{4}[f(x+1,y) + f(x-1,y) + f(x,y+1) + f(x,y-1)]\\
&\quad + \frac{1}{8}[f(x+1,y+1) + f(x-1,y+1) + f(x+1,y-1) + f(x-1,y-1)]\\
&\quad + \frac{1}{16}[f(x+2,y) + f(x-2,y) + f(x,y+2) + f(x,y-2)].
\end{aligned}
$$

Hence

$$
h_3 * h_3(x,y) = \begin{cases} \frac{1}{2} & \text{if } D(x,y) = 0,\\ \frac{1}{4} & \text{if } D(x,y) = 1,\\ \frac{1}{8} & \text{if } D(x,y) = 2,\\ \frac{1}{16} & \text{if } D(x,y) = 4,\\ 0 & \text{otherwise.} \end{cases}
$$

3. (a) i. Suppose f is a minimizer of E_1. Then for any $v : \Omega \to \mathbb{R}$,

$$\begin{aligned}
0 &= \frac{\partial}{\partial t}\bigg|_{t=0} E_1(f + tv) \\
&= \int_\Omega \frac{\partial}{\partial t}\bigg|_{t=0} [(f + tv - g)^2 + \lambda \|\nabla(f + tv)\|_2^2] \, dx \, dy \\
&= \int_\Omega [2v(f - g) + \lambda \frac{\partial}{\partial t}\bigg|_{t=0} \langle \nabla f + t\nabla v, \nabla f + t\nabla v \rangle] \, dx \, dy \\
&= \int_\Omega [2v(f - g) + \lambda \frac{\partial}{\partial t}\bigg|_{t=0} (\|\nabla f\|_2^2 + 2t\langle \nabla f, \nabla v\rangle + t^2 \|\nabla v\|_2^2)] \, dx \, dy \\
&= 2 \int_\Omega [v(f - g) + \lambda \langle \nabla f, \nabla v \rangle] \, dx \, dy \\
&= 2 \int_\Omega v[f - g - \lambda \nabla \cdot (\nabla f)] + 2\lambda \int_{\partial\Omega} v \langle \nabla f, \vec{n} \rangle \, ds. \qquad (*)
\end{aligned}$$

Since v is arbitrarily chosen, f satisfies:

$$\begin{cases} -\lambda \nabla^2 f(x,y) + f(x,y) = g(x,y) & \text{if } (x,y) \in \Omega, \\ \langle \nabla f(x,y), \vec{n}(x,y) \rangle = 0 & \text{if } (x,y) \in \partial\Omega. \end{cases}$$

ii. Suppose f satisfies the above PDE. Then for any $h : \Omega \to \mathbb{R}$,

$$\begin{aligned}
E_1(h) - E_1(f) &= \int_\Omega [(h - g)^2 - (f - g)^2 + \lambda \|\nabla h\|_2^2 - \lambda \|\nabla f\|_2^2] \, dx \, dy \\
&= \int_\Omega [h^2 - 2gh + g^2 - f^2 + 2fg - g^2 + \lambda(\|\nabla h - \nabla f\|_2^2 - 2\|\nabla f\|_2^2 + 2\langle \nabla h, \nabla f \rangle)] \\
&= \int_\Omega [(h - f)^2 + 2hf - 2f^2 - 2gh + 2fg + \lambda(\|\nabla h - \nabla f\|_2^2 + 2\langle \nabla f, \nabla h - \nabla f \rangle)] \\
&\geq 2 \int_\Omega [(f - g)(h - f) + \lambda \langle \nabla f, \nabla h - \nabla f \rangle] \, dx \, dy \\
&= 2 \int_\Omega [(f - g)(h - f) - \lambda (h - f) \nabla^2 f] \, dx \, dy + 2 \int_{\partial\Omega} (h - f) \langle \nabla f, \vec{n} \rangle \, ds = 0.
\end{aligned}$$

Hence f minimizes E_1.

iii. From $(*)$, a descent direction of E_1 at f is given by:

$$v(x,y) = \begin{cases} -f(x,y) + g(x,y) + \lambda \nabla^2 f(x,y) & \text{if } (x,y) \in \Omega, \\ -\langle \nabla f(x,y), \vec{n}(x,y) \rangle & \text{if } (x,y) \in \partial\Omega. \end{cases}$$

Hence given $f : \Omega \to \mathbb{R}$, the gradient descent scheme for minimizing E_1 is given by:

$$\frac{\partial f(x,y;t)}{\partial t} = \begin{cases} -f(x,y) + g(x,y) + \lambda \nabla^2 f(x,y) & \text{if } (x,y) \in \Omega, \\ -\langle \nabla f(x,y), \vec{n}(x,y) \rangle & \text{if } (x,y) \in \partial\Omega. \end{cases}$$

Upon discretization in time and space, the explicit gradient descent scheme with time step τ becomes:

$$\frac{f^{n+1}(x,y) - f^n(x,y)}{\tau} = \begin{cases} -f^n(x,y) + g(x,y) + \lambda \mathcal{D}_2(\mathcal{D}_1(f^n))(x,y) & \text{if } (x,y) \in \Omega, \\ -\langle \mathcal{D}_1 f^n(x,y), \vec{n}(x,y)\rangle & \text{if } (x,y) \in \partial\Omega, \end{cases}$$

where the finite difference operators $\mathcal{D}_1$ and $\mathcal{D}_2$ approximate the differential operators ∇ and $\nabla\cdot$.

iv. Using the forward difference approximations for first derivatives,

$$E_{1,discrete}(f) = \sum_{(x,y)\in\Omega} \{[f(x,y)-g(x,y)]^2 + \lambda[f(x+1,y)-f(x,y)]^2 + \lambda[f(x,y+1)-f(x,y)]^2\}.$$

v. Suppose f minimizes $E_{1,discrete}$. Then for any $(x_0, y_0) \in \Omega$,

$$\begin{aligned} 0 &= \frac{\partial E_{1,discrete}(f)}{\partial f(x_0,y_0)} \\ &= \frac{\partial}{\partial f(x_0,y_0)} \Big([f(x_0,y_0) - g(x_0,y_0)]^2 + \lambda\{[f(x_0+1,y_0) - f(x_0,y_0)]^2 + [f(x_0,y_0) - f(x_0-1,y \\ &\quad + [f(x_0,y_0+1) - f(x_0,y_0)]^2 + [f(x_0,y_0) - f(x_0,y_0-1)]^2\}\Big) \\ &= 2[f(x_0,y_0) - g(x_0,y_0)] + 2\lambda[4f(x_0,y_0) - f(x_0+1,y_0) - f(x_0-1,y_0) - f(x_0,y_0+1) \\ &\quad - f(x_0,y_0-1)], \end{aligned}$$

i.e. $(1+4\lambda)f(x_0,y_0) - \lambda f(x_0+1,y_0) - \lambda f(x_0-1,y_0) - \lambda f(x_0,y_0+1) - \lambda f(x_0,y_0-1) = g(x_0,y_0)$.

vi. The gradient of $E_{1,discrete}$ has been computed in the previous part. Hence the explicit gradient descent scheme for minimizing $E_{1,discrete}$ with time step $\tau > 0$ is given by:

$$\begin{aligned} &\frac{f^{n+1}(x,y) - f^n(x,y)}{\tau} \\ &= -2(1+4\lambda)f^n(x,y) - 2\lambda[f^n(x+1,y) + f^n(x-1,y) + f^n(x,y+1) + f^n(x,y-1)] + 2g(x,y) \end{aligned}$$

(b) i. Suppose f is a minimizer of E_2. Then for any $v : \Omega \to \mathbb{R}$,

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}\Big|_{t=0} E_2(f+tv) \\ &= \int_\Omega \frac{\partial}{\partial t}\Big|_{t=0} [(f+tv-g)^2 + K\|\nabla(f+tv)\|^4]\,dx\,dy \\ &= \int_\Omega [2v(f-g) + K\frac{\partial}{\partial t}\Big|_{t=0} (\langle \nabla f + t\nabla v, \nabla f + t\nabla v\rangle)^2]\,dx\,dy \\ &= \int_\Omega [2v(f-g) + 2K\|\nabla f\|^2 \langle \nabla f, \nabla v\rangle]\,dx\,dy \\ &= 2\int_\Omega v[f - g - \nabla\cdot(K\|\nabla f\|^2 \nabla f)]\,dx\,dy + 2\int_{\partial\Omega} vK\|\nabla f\|^2 \langle f, \vec{n}\rangle\,ds. \end{aligned}$$

Since v is arbitrarily chosen, f satisfies:

$$\begin{cases} -\nabla \cdot (K\|\nabla f\|^2 \nabla f)(x,y) + f(x,y) = g(x,y) & \text{if } (x,y) \in \Omega, \\ K(x,y)\|\nabla f(x,y)\|^2 \langle f(x,y), \vec{n}(x,y) \rangle = 0 & \text{if } (x,y) \in \partial\Omega. \end{cases}$$

iii. From i., a descent direction is given by

$$v(x,y) = \begin{cases} -f(x,y) + g(x,y) + \nabla \cdot (K\|\nabla f\|^2 \nabla f)(x,y) & \text{if } (x,y) \in \Omega, \\ -K(x,y)\|\nabla f(x,y)\|^2 \langle f(x,y), \vec{n}(x,y) \rangle & \text{if } (x,y) \in \partial\Omega. \end{cases}$$

Hence given $f : \Omega \to \mathbb{R}$, the gradient descent scheme for minimizing E_2 is given by:

$$\frac{\partial f(x,y;t)}{\partial t} = \begin{cases} -f(x,y) + g(x,y) + \nabla \cdot (K\|\nabla f\|^2 \nabla f)(x,y) & \text{if } (x,y) \in \Omega, \\ -K(x,y)\|\nabla f(x,y)\|^2 \langle f(x,y), \vec{n}(x,y) \rangle & \text{if } (x,y) \in \partial\Omega. \end{cases}$$

Upon discretization in time and space, the explicit gradient descent scheme with time step τ becomes:

$$\frac{f^{n+1}(x,y) - f^n(x,y)}{\tau} = \begin{cases} -f^n(x,y) + g(x,y) + \mathcal{D}_2(K\|D_1(f^n)\|^2 D_1(f^n))(x,y) & \text{if } (x,y) \in \Omega, \\ -K(x,y)\|\mathcal{D}_1(f)(x,y)\|^2 \langle f(x,y), \vec{n}(x,y) \rangle & \text{if } (x,y) \in \partial\Omega, \end{cases}$$

where the finite difference operators $\mathcal{D}_1$ and $\mathcal{D}_2$ approximate the differential operators ∇ and $\nabla\cdot$.