Lecture 18:

Image denoising by solving PDE (derived from energy minimisation problem)

Consider the harmonic-L2 minimization model:

$$\text{minimize} \quad E(f) = \int_{\underbrace{\Omega}_{\text{image domain}}} (f(x,y) - \underbrace{g(x,y)}_{\text{Observed}})^2 \, dx\,dy + \int \underbrace{|\nabla f|}_{\text{smoothness of } f}^2 \, dx\,dy$$

(Look for (continuous) image f)

We find f that minimizes $E(f)$.

Take any function $v(x,y)$. Consider a real-valued function $S: \mathbb{R} \to \mathbb{R}$:

$$S(\varepsilon) := E(f + \varepsilon v) = \int_\Omega \left(f(x,y) + \varepsilon v(x,y) - g(x,y)\right)^2 dx\,dy + \int_\Omega |\nabla f + \varepsilon \nabla v|^2 \, dx\,dy$$

$$\frac{d}{d\varepsilon} S(\varepsilon) = 2\int_\Omega \left(f(x,y) + \varepsilon v(x,y) - g(x,y)\right)^2 dx\,dy + 2\int_\Omega \left[\left(\frac{\partial f}{\partial x} + \varepsilon \frac{\partial v}{\partial x}\right)\frac{\partial v}{\partial x} + \left(\frac{\partial f}{\partial y} + \varepsilon \frac{\partial v}{\partial y}\right)\frac{\partial v}{\partial y}\right] dx\,dy$$

If f is a minimizer, $\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} S(\varepsilon) = 0$ for all v ($\because$ $S(0)$ is the minimum)

$$\therefore\ S'(0) = 0 = 2\int_\Omega \big(f(x,y) - g(x,y)\big)\, v(x,y)\, dx\,dy + 2\int_\Omega (f_x v_x + f_y v_y)\, dx\,dy \quad \text{for all } v.$$

Remark: If we can formulate the above equation as follows:

$$\int_\Omega T(x,y)\, v(x,y)\, dx\,dy = 0 \quad \text{for all } v(x,y)$$

then, we can conclude $T(x,y) = 0$ in Ω.

In our case, first term is okay!
second term NOT okay!

Useful Tool: (Integration by part)

$$\nabla\cdot\big(V_1(x,y), V_2(x,y)\big) = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y}$$

$$\int_\Omega \nabla f \cdot \nabla g\, dx\,dy = -\int_\Omega \big(\overbrace{\nabla\cdot}^{\text{divergence}}(\nabla f)\big)\, g\, dx\,dy + \int_{\partial\Omega} g\, (\nabla f \cdot \vec{n})\, ds$$

where $\vec{n} = (n_1, n_2)$ = outward normal on the boundary.

In our case, we get: $0 = \int_\Omega (f-g)\, v\, dx\,dy - \int_\Omega (\nabla\cdot\nabla f)\, v\, dx\,dy + \int_{\partial\Omega} (\nabla f\cdot\vec{n})\, ds$

Overall, we get : $\int_{\Omega} (f - g - \Delta f)\, v \, dx\, dy - \int_{\partial\Omega} (\nabla f \cdot \vec{n})\, v \, ds = 0$ for all v

We conclude : $\begin{cases} f - g - \Delta f = 0 & \text{in } \Omega \\ \nabla f \cdot \vec{n} = 0 & \text{in } \partial\Omega \end{cases}$ (PDE)

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Remark: • Anisotropic diffussion is related to minimizing:

$$E(f) = \int_\Omega k(x,y)\,|\nabla f(x,y)|^2\,dx\,dy$$

• Energy minimization approach for solving imaging problem is called the variational image processing!

$$E(f+\varepsilon v) = \int k(x,y)\,|\nabla f + \varepsilon \nabla v|^2\,dx\,dy$$

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} E(f+\varepsilon v) = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \int_\Omega k(x,y)\,(\nabla f + \varepsilon \nabla v)\cdot(\nabla f + \varepsilon \nabla v)$$

$$= \int_\Omega k(x,y)\,(2\nabla f\cdot \nabla v)$$

$$= -\int_\Omega \left(\nabla\cdot\left(k(x,y)\nabla f\right)\right) v + \int_{\partial\Omega} \left(k(x,y)\nabla f\cdot \vec{n}\right) v$$

Total variation (TV) denoising (ROF)

Invented by: Rudin, Osher, Fatemi

Motivation: Previous model: $f = g + \Delta f$. Solve for f from noisy g.

Disadvantage: smooth out edge.

Modification: $f = g + \nabla \cdot (K \nabla f)$

K is small on edges!!

Goal: Given a noisy image $g(x,y)$, we look for $f(x,y)$ that solves:

$$f = g + \lambda \frac{\partial}{\partial x}\left(\frac{1}{|\nabla f|(x,y)} \frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{|\nabla f|(x,y)} \frac{\partial f}{\partial y}\right) \qquad (*)$$

Remark: Problem arises if $|\nabla f(x,y)| = 0$. Take care of it later.

We'll show that $(*)$ must be satisfied by a minimizer of:

$$J(f) = \frac{1}{2}\int_\Omega (f(x,y) - g(x,y))^2 + \underbrace{\lambda}_{\text{constant parameter} > 0} \int_\Omega |\nabla f(x,y)| \, dx\, dy$$

Same idea: Let $S(\varepsilon) := E(f+\varepsilon v)$

$$= \frac{1}{2}\int_\Omega (f+\varepsilon v - g)^2 + \lambda \int_\Omega \underbrace{|\nabla f + \varepsilon \nabla v|}_{\sqrt{(\nabla f + \varepsilon \nabla v)\cdot(\nabla f + \varepsilon \nabla v)}}$$

$$\frac{d}{d\varepsilon} S(\varepsilon) = \left[\int_\Omega (f+\varepsilon v - g) v + \lambda \int_\Omega \frac{\nabla f \cdot \nabla v + 2\varepsilon \nabla v \cdot \nabla v}{\sqrt{(\nabla f + \varepsilon \nabla v)\cdot(\nabla f + \varepsilon \nabla v)}} \right]$$

If f is a minimizer, $\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} S(\varepsilon) = 0$ for all v.

$$\therefore\ S'(0) = 0 = \int_\Omega (f-g)v + \lambda \int_\Omega \frac{\nabla f \cdot \nabla v}{|\nabla f|}$$

$$= \int_\Omega (f-g)v - \lambda \int_\Omega \nabla \cdot \left(\frac{\nabla f}{|\nabla f|}\right) v + \lambda \int_{\partial\Omega} \left(\frac{\nabla f}{|\nabla f|} \cdot \vec{n}\right) v$$

$$= \int_\Omega \left[(f-g) - \lambda \nabla \cdot \left(\frac{\nabla f}{|\nabla f|}\right) \right] v + \lambda \int_{\partial\Omega} \left(\frac{\nabla f}{|\nabla f|} \cdot \vec{n}\right) v \quad \text{for all } v$$

We conclude: $(f-g) - \lambda \nabla \cdot \left(\frac{\nabla f}{|\nabla f|}\right) = 0$!!

In the discrete case,

$$J(f) = \frac{1}{2}\sum_{x=1}^{N}\sum_{y=1}^{N}(f(x,y)-g(x,y))^2+\lambda\sum_{x=1}^{N}\sum_{y=1}^{N}\sqrt{(f(x+1,y)-f(x,y))^2+(f(x,y+1)-f(x,y))^2}$$

J can be regarded as a multi-variable function depending on :

$f(1,1), f(1,2), \ldots, f(1,N), f(2,1), \ldots, f(2,N), \ldots, f(N,N)$.

If f is a minimizer, then $\frac{\partial J}{\partial f(x,y)} = 0$ for all (x,y).

$$\begin{aligned}\frac{\partial J}{\partial f(x,y)} &= (f(x,y)-g(x,y)) + \lambda\frac{2(f(x+1,y)-f(x,y))(-1)+2(f(x,y+1)-f(x,y))(-1)}{2\sqrt{(f(x+1,y)-f(x,y))^2+(f(x,y+1)-f(x,y))^2}} \\ &+ \lambda\frac{2(f(x,y)-f(x-1,y))}{2\sqrt{(f(x,y)-f(x-1,y))^2+(f(x-1,y+1)-f(x-1,y))^2}} \\ &+ \lambda\frac{2(f(x,y)-f(x,y-1))}{2\sqrt{(f(x+1,y-1)-f(x,y-1))^2+(f(x,y)-f(x,y-1))^2}} = 0\end{aligned}$$

By simplification:

$$
\begin{aligned}
f^{n+1}(x,y) - g(x,y) = \lambda \Bigg\{ & \frac{f^{n}(x+1,y) - f(x,y)}{\sqrt{(f(x+1,y)-f(x,y))^2 + (f(x,y+1)-f(x,y))^2}} \\
& - \frac{f(x,y) - f(x-1,y)}{\sqrt{(f(x,y)-f(x-1,y))^2 + (f(x-1,y+1)-f(x-1,y))^2}} \Bigg\} \\
+ \lambda \Bigg\{ & \frac{f(x,y+1) - f(x,y)}{\sqrt{(f(x+1,y)-f(x,y))^2 + (f(x,y+1)-f(x,y))^2}} \\
& - \frac{f(x,y) - f(x,y-1)}{\sqrt{(f(x+1,y-1)-f(x,y-1))^2 + (f(x,y)-f(x,y-1))^2}} \Bigg\}
\end{aligned}
$$

$$= \frac{\left.\frac{\partial f}{\partial x}\right|_{(x,y)}}{\left|\nabla f\right|_{(x,y)}}$$

$$= \frac{\left.\frac{\partial f}{\partial x}\right|_{(x-1,y)}}{\left|\nabla f\right|_{(x-1,y)}}$$

$$= \frac{\left.\frac{\partial f}{\partial y}\right|_{(x,y)}}{\left|\nabla f\right|_{(x,y)}}$$

$$= \frac{\left.\frac{\partial f}{\partial y}\right|_{(x,y-1)}}{\left|\nabla f\right|_{(x,y-1)}}$$

Discretization of $f - g = \lambda \, \nabla \cdot \left(\frac{\nabla f}{|\nabla f|} \right)$