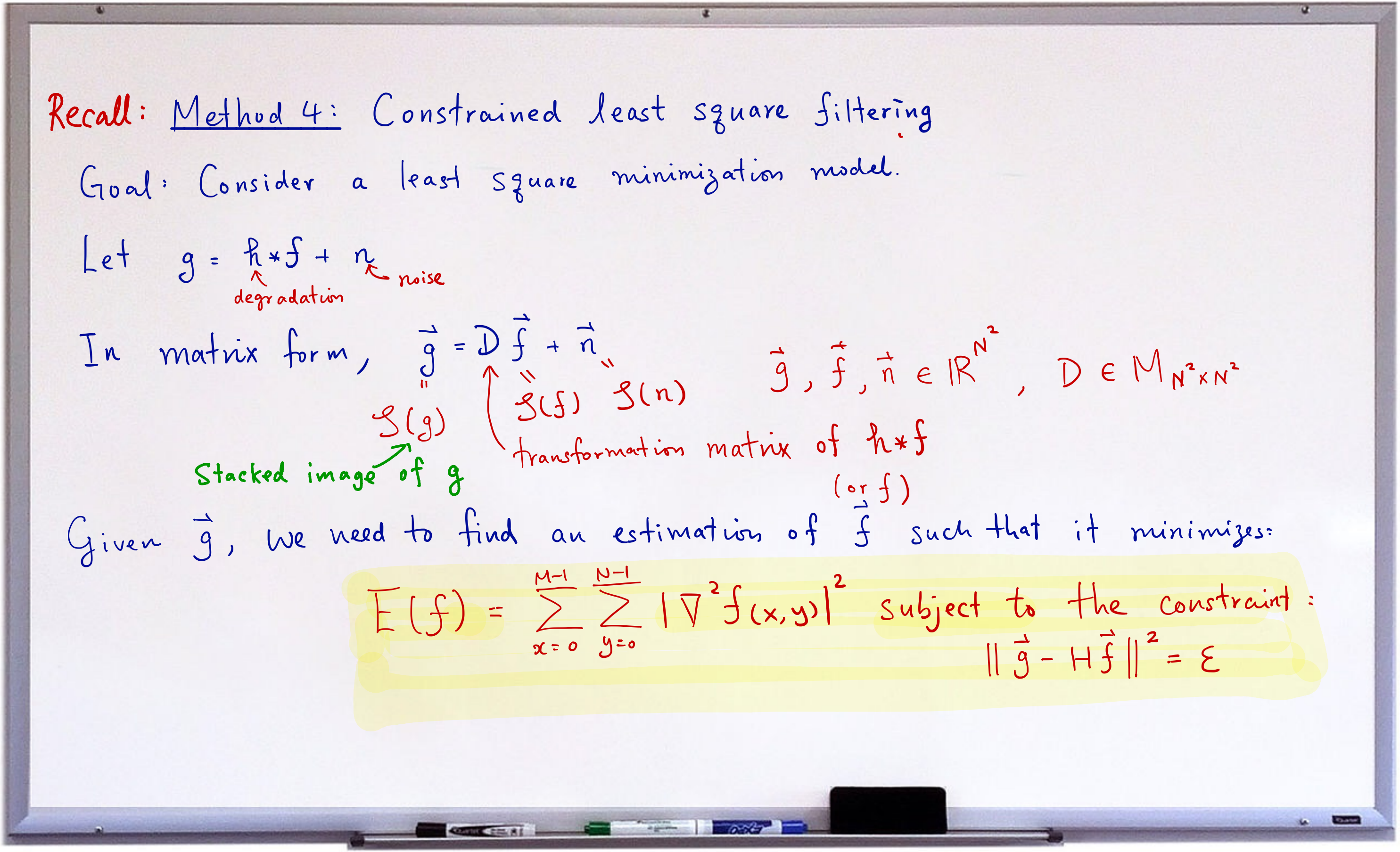

Recall: <u>Method 4:</u> Constrained least square filtering

Goal: Consider a least square minimization model.

Let $g = \underset{\text{degradation}}{h} * f + \underset{\text{noise}}{n}$

In matrix form, $\vec{g} = D\vec{f} + \vec{n}$

$\vec{g} = \mathcal{S}(g)$, $\vec{f} = \mathcal{S}(f)$, $\vec{n} = \mathcal{S}(n)$

$\vec{g}, \vec{f}, \vec{n} \in \mathbb{R}^{N^2}$, $D \in M_{N^2 \times N^2}$

Stacked image of g

transformation matrix of $h * f$ (or $\vec{f}$)

Given $\vec{g}$, we need to find an estimation of $\vec{f}$ such that it minimizes:

$$E(f) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} |\nabla^2 f(x,y)|^2 \quad \text{subject to the constraint:}$$

$$\|\vec{g} - H\vec{f}\|^2 = \varepsilon$$

- $\displaystyle\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} |\nabla^2 f(x,y)|^2$ ← Denoise
- $\|\vec{g} - D\vec{f}\|^2 = \varepsilon$ ← Deblur

In the discrete case, we can estimate:

$$\nabla^2 f(x,y) \approx f(x+1,y) + f(x,y+1) + f(x-1,y) + f(x,y-1) - 4f(x,y)$$

Taylor expansion:

$$\frac{\partial^2 f(x,y)}{\partial x^2} \approx \frac{f(x+h,y) - 2f(x,y) + f(x-h,y)}{h^2} \xrightarrow{\text{Put } h=1} \nabla^2 f(x,y) = \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)(x,y)$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) \approx \frac{f(x,y+h) - 2f(x,y) + f(x,y-h)}{h^2}$$

More generally, $\nabla^2 f = p * f$ ← discrete convolution

where

$$p = \begin{array}{cc} \left(\begin{array}{ccccc} 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & & 1 & & \vdots \\ \vdots & 1 & -4 & 1 & \vdots \\ \vdots & & 1 & & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{array}\right) & x=0 \\ y=0 & \end{array}$$

Remark: $\underset{\|\vec{n}\|^2}{\underset{\shortparallel}{\|\vec{g} - D\vec{f}\|^2}} = \varepsilon$ means we allow some fixed level of noise.

Assume $S(p * f) = \underset{\uparrow}{L}\vec{f}$ transformation matrix representing the convolution with p.

Then: $E(\vec{f}) = (L\vec{f})^T (L\vec{f})$

Theorem: The constrained least square problem has the optimal solution in the spatial domain that satisfies:

$$(D^T D + \gamma L^T L)\vec{f} = D^T \vec{g}$$

for some suitable parameter γ.

In the frequency domain,

$$\hat{F}(u,v) := DFT(f)(u,v) = \frac{1}{N^2} \frac{\overline{H(u,v)}}{|H(u,v)|^2 + \gamma |P(u,v)|^2} G(u,v)$$

$\Big(H = DFT(h)$; $G(u,v) = DFT(g)$; $P(u,v) = DFT(p)$ where

$$p = \begin{pmatrix} 0 & \cdots & & \cdots & 0 \\ \vdots & & 1 & & \vdots \\ & 1 & -4 & 1 & \\ \vdots & & 1 & & \vdots \\ 0 & \cdots & & \cdots & 0 \end{pmatrix} \Big)$$

Remark: Constrained least square filtering:

$$T(u,v) = \frac{1}{N^2} \frac{\overline{H(u,v)}}{|H(u,v)|^2 + \gamma |P(u,v)|^2}$$

Let $\tilde{F}(u,v) = T(u,v)\, G(u,v)$

Compute Inverse DFT of $\tilde{F}(u,v)$.

Eq$\dagger$ in spatial domain is proved (Last time). We'll consider the frequency domain.

Note that D and L are transformation matrix of convolution.

$\therefore$ D and L are block-circulant.

Some facts about circulant matrix:

Recall: A matrix is block-circulant if

$$H = \begin{pmatrix} H_0 & H_{M-1} & H_{M-2} & \cdots & H_1 \\ H_1 & H_0 & H_{M-1} & \cdots & H_2 \\ H_2 & H_1 & H_0 & \cdots & H_3 \\ \vdots & \vdots & \vdots & & \vdots \\ H_{M-1} & H_{M-2} & H_{M-3} & \cdots & H_0 \end{pmatrix}$$

(each H_i is circulant)

A matrix $\mathcal{C}$ is circulant if:

$$\mathcal{C} = \begin{pmatrix} d_0 & d_{M-1} & d_{M-2} & \cdots & d_1 \\ d_1 & d_0 & d_{M-1} & \cdots & d_2 \\ d_2 & d_1 & d_0 & \cdots & d_3 \\ \vdots & \vdots & \vdots & & \vdots \\ d_{M-1} & d_{M-2} & d_{M-3} & \cdots & d_0 \end{pmatrix}$$

Eigenvalues / Eigenvectors of circulant $\mathcal{C}$

Let $\mathcal{C} = \begin{pmatrix} d(0) & d(M-1) & \cdots & d(1) \\ d(1) & d(0) & \cdots & d(2) \\ \vdots & \vdots & \cdots & \vdots \\ d(M-1) & d(M-2) & \cdots & d(0) \end{pmatrix}$ be a circulant matrix. Then the eigenvalues of $\mathcal{C}$ is given by:

$$\underline{\lambda(k) = d(0) + d(1)e^{\frac{2\pi j}{M}(M-1)k} + d(2)e^{\frac{2\pi j}{M}(M-2)k} + \cdots + d(M-1)e^{\frac{2\pi j}{M}k}}$$

(eigenvalue)

where $k = 0, 1, 2, \cdots M-1$.

Its associated eigenvector is given by:

$$\boxed{\vec{w}(k) = \begin{pmatrix} 1 \\ e^{\frac{2\pi j}{M}k} \\ e^{\frac{2\pi j}{M}2k} \\ \vdots \\ e^{\frac{2\pi j}{M}(M-1)k} \end{pmatrix}}$$

(eigenvector)

Using the fact that both D and L are block-circulant, we can check that:

Fact 1:

$$D = W\Lambda_D W^{-1}, D^T = W\Lambda_D^* W^{-1}, L = W\Lambda_L W^{-1}, L^T = W\Lambda_L^* W^{-1}$$

where W is invertible and Λ_D, Λ_L are diagonal matrices.

Also,

$$\Lambda_D(k,i) = \begin{cases} N^2 H\left(\mathrm{mod}_N(k), \left\lfloor \frac{k}{N} \right\rfloor\right) & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$$

DFT(h)

where $H = DFT(h)$.

and

$$\Lambda_L(k,i) = \begin{cases} N^2 P\left(\mathrm{mod}_N(k), \left\lfloor \frac{k}{N} \right\rfloor\right) & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$$

$P = DFT(p)$;

$$p = \begin{pmatrix} 0 & \cdots & & \cdots & 0 \\ \vdots & & 1 & & \vdots \\ \vdots & 1 & -4 & 1 & \vdots \\ \vdots & & 1 & & \vdots \\ 0 & \cdots & & \cdots & 0 \end{pmatrix}$$

$$\Lambda_D = \begin{pmatrix} N^2 H(0,0) & & & & & & \\ & N^2 H(1,0) & & & & & \\ & & \ddots & & & & \\ & & & N^2 H(N-1,0) & & & \\ & & & & N^2 H(1,1) & & \\ & & & & & \ddots & \\ & & & & & & N^2 H(N-1,1) \\ & & & & & & & \ddots \end{pmatrix}$$

1st col of H becomes first N diagonal entries

2nd col of H becomes 2nd N diagonal entries

What is W ?

Let $W_N = \left(\frac{1}{\sqrt{N}} e^{\frac{2\pi j}{N} k n} \right)_{0 \leq k, n \leq N-1} \in M_{N \times N}(\mathbb{C})$

Then:

$$W = W_N \otimes W_N \in M_{N^2 \times N^2}(\mathbb{C})$$

$$A \otimes B = \begin{pmatrix} a_{11} B & a_{12} B & \cdots & a_{N1} B \\ \vdots & \ddots & & \vdots \\ a_{N1} B & a_{N2} B & \cdots & a_{NN} B \end{pmatrix} \in M_{N^2 \times N^2}(\mathbb{C})$$

And we can check that $W^{-1} = W_N^{-1} \otimes W_N^{-1}$

Suppose D is the transformation matrix representing the convolution with h.

(In other words, if $g = h * f$, then: $\underset{\mathbb{R}^{N^2}}{\overset{}{\underset{\cap}{\vec{g}}}} = \underset{M_{N^2 \times N^2}}{\underset{\cap}{D}}\,\vec{f}_{\in \mathbb{R}^{N^2}}$)

Let $H = \mathrm{DFT}(h) \in M_{N \times N}$

Diagonalization of D:

$$D = WN^2 \underbrace{\left[\begin{pmatrix} H(0,0) &&&&&&&&&& \\ & H(1,0) &&&&&&&&& \\ && \ddots &&&&&&&& \\ &&& H(N-1,0) &&&&&&& \\ &&&& H(0,1) &&&&&& \\ &&&&& \ddots &&&&& \\ &&&&&& H(N-1,1) &&&& \\ &&&&&&& \ddots &&& \\ &&&&&&&& H(0,N-1) && \\ &&&&&&&&& \ddots & \\ &&&&&&&&&& H(N-1,N-1) \end{pmatrix}\right]}_{\text{Stack } H \text{ to form the diagonal matrix.}} W^{-1}$$

Fact 2:

$$W^{-1}\vec{f} = N\begin{pmatrix} F(0,0) \\ \vdots \\ F(N-1,0) \\ F(0,1) \\ \vdots \\ F(N-1,1) \\ \vdots \\ F(0,N-1) \\ \vdots \\ F(N-1,N-1) \end{pmatrix} = N\mathcal{S}(F) \quad \text{where } F = DFT(f).$$

Combining these information and substitute into the "governing" equation :

$$(D^T D + \gamma L^T L)\vec{f} = D^T \vec{g},$$

We get :

$$\cancel{W}(\Lambda_D^* \Lambda_D + \gamma \Lambda_L^* \Lambda_L) \underbrace{W^{-1}\vec{f}}_{\mathcal{F}(F)} = \cancel{W}\Lambda_D^* \underbrace{W^{-1}\vec{g}}_{\mathcal{F}(G)}$$

Combining these information and substitute into the "governing" equation :

$$(D^T D + \gamma L^T L)\vec{f} = D^T \vec{g} ,$$

We get:

$$\cancel{W}(\Lambda_D^* \Lambda_D + \gamma \Lambda_L^* \Lambda_L) W^{-1}\vec{f} = \cancel{W}\Lambda_D^* W^{-1}\vec{g}$$

We can check that:

①

$$\boxed{\Lambda_D^* \Lambda_D = \begin{pmatrix} N^4|H(0,0)|^2 & & & & \\ & N^4|H(1,0)|^2 & & & \\ & & \ddots & & \\ & & & N^4|H(N-1,0)|^2 & \\ & & & & \ddots & \\ & & & & & N^4|H(N-1,N-1)|^2 \end{pmatrix}}$$

②

$$\boxed{\Lambda_L^* \Lambda_L = \begin{pmatrix} N^4|P(0,0)|^2 & & & & \\ & N^4|P(1,0)|^2 & & & \\ & & \ddots & & \\ & & & N^4|P(N-1,0)|^2 & \\ & & & & \ddots & \\ & & & & & N^4|P(N-1,N-1)|^2 \end{pmatrix}}$$

③

$$\boxed{W^{-1}\vec{f} = N\mathcal{S}(F), W^{-1}\vec{g} = N\mathcal{S}(G)} \text{ where } F = DFT(f), G = DFT(g).$$

Combining these information and substitute into the "governing" equation :

$$\left(D^T D + \gamma L^T L\right)\vec{f} = D^T \vec{g} ,$$

We get:

$$\cancel{W}(\Lambda_D^* \Lambda_D + \gamma \Lambda_L^* \Lambda_L) W^{-1}\vec{f} = \cancel{W}\Lambda_D^* W^{-1}\vec{g}$$

$$\therefore \; N^4 \begin{pmatrix} |H(0,0)|^2+\gamma\,|P(0,0)|^2 & & & & \\ & |H(1,0)|^2+\gamma\,|P(1,0)|^2 & & & \\ & & \ddots & & \\ & & & |H(N-1,0)|^2+\gamma\, P(N-1,0)|^2 & \\ & & & & \ddots \end{pmatrix} N \begin{pmatrix} F(0,0) \\ F(1,0) \\ \vdots \\ F(N-1,0) \\ \vdots \end{pmatrix} =$$

$$N^2 \begin{pmatrix} \overline{H(0,0)} & & & & \\ & \overline{H(1,0)} & & & \\ & & \ddots & & \\ & & & \overline{H(N-1,0)} & \\ & & & & \ddots \end{pmatrix} N \begin{pmatrix} G(0,0) \\ G(1,0) \\ \vdots \\ G(N-1,0) \\ \vdots \end{pmatrix}$$

Combining all these, we get for every (u,v),

$$N^4[|H(u,v)|^2 + \gamma|P(u,v)|^2]NF(u,v) = N^2\overline{H(u,v)}NG(u,v)$$

$$\Rightarrow \boxed{N^2 \frac{|H(u,v)|^2 + \gamma|P(u,v)|^2}{\overline{H(u,v)}} F(u,v) = G(u,v)}$$

Summary: Constrained least square filtering minimizes:

$$E(\vec{f}) = (L\vec{f})^T(L\vec{f})$$

Subject to the constraint that:

$$\|\underbrace{\vec{g} - H\vec{f}}_{\vec{n}}\|^2 = \varepsilon$$

(allow fixed amount of noise)

Diagonalization of block-circulant matrix H

Let H be the block-circulant matrix as defined above. Define a matrix with elements:

$$W_N(k,n) := \frac{1}{\sqrt{N}} exp\left(\frac{2\pi j}{N} kn\right) \qquad 0 \leq n \leq N-1$$

Consider the **Kronecker product** $\otimes$ of W_N with itself:

$$W := W_N \otimes W_N$$

Recall that: Kronecker product is defined as:

The **Kronecker product** of two matrices are given by:

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1N}B \\ a_{21}B & a_{22}B & \cdots & a_{2N}B \\ \vdots & \vdots & & \vdots \\ a_{N1}B & a_{N2}B & \cdots & a_{NN}B \end{pmatrix}$$

$$A = (a_{ij})_{0 \leq i,j \leq N-1}$$

$$B = (b_{ij})_{0 \leq i,j \leq N-1}$$

W^{-1} can be easily computed!

Easy to check: $W^{-1} = W_N^{-1} \otimes W_N^{-1}$ where:

$$W_N^{-1}(k,n) := \frac{1}{\sqrt{N}} exp\left(-\frac{2\pi j}{N} kn\right) \qquad 0 \le n \le N-1$$

Let

$$\Lambda(k,i) = \begin{cases} N^2 \ddot{H}\left(\mathrm{mod}_N(k), \left\lfloor \frac{k}{N} \right\rfloor\right) & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$$

where H = DFT of the point spread function h, $\left\lfloor \frac{k}{N} \right\rfloor$ = largest integer smaller than or equal to $\frac{k}{N}$ and $\mathrm{mod}_N(k) = k(\mathrm{mod}\ N)$ (e.g. $10(\mathrm{mod}\ 3) = 1$)

Then, we can show that $H = W\Lambda W^{-1}$ and $H^{-1} = W\Lambda^{-1}W^{-1}$.

Also, $H^T = W\Lambda^* W^{-1}$. (Λ^* is the complex conjugate of Λ)

By direct calculation, it is easy to check that $W^{-1}\vec{g} = N\zeta(G)$ where $G = DFT(g)$.

Example : Assume that :

$$G = \begin{pmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{pmatrix} \quad \text{and} \quad W_3^{-1} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & exp\left(-\frac{2\pi j}{3}\right) & exp\left(-\frac{2\pi j}{3}2\right) \\ 1 & exp\left(-\frac{2\pi j}{3}2\right) & exp\left(-\frac{2\pi j}{3}\right) \end{pmatrix}$$

Then:

$$W^{-1} = W_3^{-1} \otimes W_3^{-1} = \frac{1}{3} \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} \\
1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} \\
1 & 1 & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} \\
1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}3} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}3} & e^{-\frac{2\pi j}{3}4} \\
1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}3} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}4} & e^{-\frac{2\pi j}{3}3} \\
1 & 1 & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} \\
1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}3} & e^{-\frac{2\pi j}{3}4} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}3} \\
1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}4} & e^{-\frac{2\pi j}{3}3} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}3} & e^{-\frac{2\pi j}{3}2}
\end{pmatrix}$$

$$W^{-1}\vec{g} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} \\ 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} \\ 1 & 1 & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} \\ 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}} \\ 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & 1 \\ 1 & 1 & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} \\ 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & 1 \\ 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}2} \end{pmatrix}\begin{pmatrix} g_{00} \\ g_{10} \\ g_{20} \\ g_{01} \\ g_{11} \\ g_{21} \\ g_{02} \\ g_{12} \\ g_{22} \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} g_{00} + g_{10} + g_{20} + g_{01} + g_{11} + g_{21} + g_{02} + g_{12} + g_{22} \\ g_{00} + g_{10}e^{-\frac{2\pi j}{3}} + g_{20}e^{-\frac{2\pi j}{3}2} + g_{01} + g_{11}e^{-\frac{2\pi j}{3}} + g_{21}e^{-\frac{2\pi j}{3}2} + g_{02} + g_{12}e^{-\frac{2\pi j}{3}} + g_{22}e^{-\frac{2\pi j}{3}2} \\ \vdots \\ \vdots \\ \vdots \end{pmatrix}$$

$= 3^2 G(0,0)$

$= 3^2 G(1,0)$

$\vdots$

$G = \mathrm{DFT}(g)$

$\therefore \quad W^{-1}\vec{g} = 3\,\mathcal{S}(G)$

Image sharpening in the frequency domain

Goal: Enhance image so that it shows more obvious edges.

Method 1: Laplacian masking

Recall that : $\Delta f(x,y) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$.

In the discrete case, $\Delta f(x,y) \approx f(x+1, y) + f(x,y+1) + f(x,y-1) + f(x-1,y) - 4f(x,y)$

or $\Delta f \approx p * f$ where $p = \begin{pmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{pmatrix}$

We can observe that $-\Delta f$ captures the edges of the image (leaving other region zero)

$$\therefore \text{Shapen image} = f + \overbrace{(-\Delta f)}^{\text{add more edges}}$$

In the frequency domain: $DFT(g) = DFT(f) - DFT(\overset{p*f}{\overset{\shortparallel}{\Delta f}})$

$= DFT(f) - c\,DFT(p).DFT(f)$

$$\therefore DFT(g) = [1 - \underset{c\,DFT(p)}{\underset{\shortparallel}{H_{laplacian}}}(u,v)]\, DFT(f)(u,v)$$