1. This question help you review the definition of low/high pass filters in the frequency domain. Please refer to lecture notes first if you are not familiar with them. If you are very familiar with them, you can skip this question.

   (a) Assume central spectrum, apply

      i. the ideal low-pass filter of radius 2;
      ii. the ideal high-pass filter of radius 2;
      iii. the Butterworth low-pass filter of radius 2 and order 2;
      iv. the Butterworth high-pass filter of radius 2 and order 2;
      v. the Gaussian low-pass filter of spread 2;
      vi. the Gaussian high-pass filter of spread 2;

      to the following 5×5 matrices:

      i. $f_1 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$;

      ii. $f_2 = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 \end{pmatrix}$;

   (b) Without assuming central spectrum, i.e., the matrices are indexed by $\{0, \cdots, M-1\} \times \{0, \cdots, N-1\}$, apply

      i. the ideal low-pass filter of radius 3;
      ii. the ideal high-pass filter of radius 3;
      iii. the Butterworth low-pass filter of radius 2 and order 2;
      iv. the Butterworth high-pass filter of radius 2 and order 2;
      v. the Gaussian low-pass filter of spread 2;
      vi. the Gaussian high-pass filter of spread 2;

      to the following 5×5 matrices:

      i. $f_1 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$;

      ii. $f_2 = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & -1 \end{pmatrix}$;

2. This question help you review the definition of image enhancement in the frequency domain. Please refer to lecture notes first if you are not familiar with them. If you are very familiar with them, you can skip this question.

   (a) Perform Laplacian mask to the following image $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$;

(b) Perform unsharp masking Using the following low-pass filters:

i. ideal low-pass filter of radius 2,

ii. Butterworth low-pass filter of radius 2 and order 2,

iii. Gaussian low-pass filter of spread 2,

on the following 4×4 images:

i. $f_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$;

ii. $f_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$;

3. This question is in a more abstract setting.

(a) Consider a $2N \times 2N$ image $I = (I(m,n))_{-N \le m,n \le N-1}$. The Butterworth high-pass filter H of squared radius D_0 and order n is applied on $DFT(I) = (\hat{I}(u,v))_{-N \le u,v \le N-1}$ to give $G(u,v)$. Suppose $\hat{I}(2,1) \neq 0$ and $\hat{I}(-1,3) \neq 0$, and

$$G(2,1) = \frac{25}{26}\hat{I}(2,1) \text{ and } G(-1,3) = \frac{100}{101}\hat{I}(-1,3).$$

Find D_0 and n.

(b) Consider a $(2M+1) \times (2N+1)$ image $I = (I(m,n))_{-M \le m \le M,\, -N \le n \le N}$, where $M, N > 200$. Suppose we perform unsharp masking $H(u,v)$ using Gaussian high-pass filter with standard deviation σ to $DFT(I) = (\hat{I}(u,v))_{-M \le u \le M,\, -N \le v \le N}$. Suppose $H(2,2) = \dfrac{MN+1}{MN}$. Find σ^2.

4. Compute the degradation functions in the frequency domain that correspond to the following $M \times N$ convolution kernels h, i.e. find $H \in M_{M\times N}(\mathbb{C})$ such that

$$DFT(h * f)(u,v) = H(u,v)DFT(f)(u,v)$$

for any periodically extended $f \in M_{M\times N}(\mathbb{R})$:

(a) Assuming integer k satisfies $k \le \min\{\frac{M}{2}, \frac{N}{2}\}$,

$$h_1(x,y) = \begin{cases} \frac{1}{(2k+1)^2} & \text{if } dist(x, M\mathbb{Z}) \le k \text{ and } dist(y, N\mathbb{Z}) \le k, \\ 0 & \text{otherwise;} \end{cases}$$

(b)

$$h_4(x,y) = \begin{cases} -4 & \text{if } D(x,y) = 0, \\ 1 & \text{if } D(x,y) = 1, \\ 0 & \text{otherwise;} \end{cases}$$

(c) Letting $a, b \in \mathbb{Z}$ and $T \in \mathbb{N} \setminus \{0\}$ such that $|a|(T-1) < M$ and $|b|(T-1) < N$,

$$h_5(x,y) = \begin{cases} \frac{1}{T} & \text{if } (x,y) \in \{(at, bt) : t = 0, 1, \cdots, T-1\} \\ 0 & \text{otherwise.} \end{cases}$$

5. Prove that for any $f \in M_{M\times N}(\mathbb{C})$,

$$\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} |DFT(f)(m,n)|^2 = \frac{1}{MN}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1} |f(k,l)|^2 .$$

6. This question help you review the definition of filters in the spatial domain. Please refer to lecture notes first if you are not familiar with them. If you are very familiar with them, you can skip this question. Apply

   i. the mean filter of size 3×3;

   ii. the Gaussian filter of spread 2;

   iii. the median filter size 3×3;

   to the following 5×5 images:

   (a) $f_1 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$;

   (b) $f_2 = \begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 & 1 \end{pmatrix}$;

7. Find $g_i \in \mathbb{M}_{N\times N}(\mathbb{R})$, $N \geq 5$, such that $g_i * f = h_i * (h_i * f)$ for any $f \in M_{N\times N}(\mathbb{R})$:

   (a) $h_1(x,y) = \begin{cases} 1 & \text{if } (x,y) = (-1,0), \\ -1 & \text{if } (x,y) = (0,0), \\ 0 & \text{otherwise;} \end{cases}$

   (b) $h_2(x,y) = \begin{cases} \frac{1}{9} & \text{if } D(x,y) \leq 2, \\ 0 & \text{otherwise;} \end{cases}$

   (c) $h_3(x,y) = \begin{cases} \frac{1}{2} & \text{if } D(x,y) = 0, \\ \frac{1}{4} & \text{if } D(x,y) = 1, \\ 0 & \text{otherwise.} \end{cases}$

   (d) Explain how to find filters H_i in the frequency domain such that $H_i \odot DFT(f) = DFT(g_i * f)$ for (a), (b), and (c).

8. For the following energy functionals, first find the PDEs that must be satisfied by the minimizers of them, and then derive a iterative scheme to compute the minimizers in both the continuous and the discrete setting

   (a) $E_1(f) = \int_\Omega [(f-g)^2 + K\|\nabla f\|^4]\, dx\, dy$, $K(x,y) > 0$ non-constant;

   (b) $E_2(f) = \int_\Omega \sum_{i=1}^{N} (f-g_i)^2 + \lambda\|\nabla f\|^2 dx\, dy$, where $g_i, i = 1, \cdots, N$ are observations of noisy images.

   (c) $E_3(f) = \int_\Omega [(h * f - g)^2 + K\|\nabla f\|]\, dx\, dy$, $K(x,y) > 0$ non-constant.

9. Using the active contour model, we consider the following energy:

$$E(\gamma) = \int_0^{2\pi} |\gamma'(s)|^2 \ + \lambda|\gamma''(s)|^2 + \mu V(\gamma(s)) \ ds$$

where $\mu, \lambda > 0$ and the edge detector V is defined as

$$V(x,y) = \begin{cases} x^2 + y^2 & \text{if } \ \mathrm{x}^2 + y^2 \geq 1 \\ \dfrac{1}{x^2+\epsilon} + \dfrac{1}{y^2+\epsilon} & \text{if } \ \mathrm{x}^2 + y^2 < 1 \end{cases}$$

Find the differential equation that must be satisfied by the minimizer. Also, formulate the energy in the discrete setting and propose an iterative scheme to find the minimizer.