MATH3360: Mathematical Imaging

Assignment 4

Due: 17 December 2021

Please give detailed steps and reasons in your solutions.

1. Given $N^2 \times N^2$ block-circulant real matrices D and L, $N \times N$ image g and fixed parameter $\varepsilon > 0$, the constrained least square filtering aims to find $f \in M_{N\times N}$ that minimizes:

$$E(f) = [L\mathcal{S}(f)]^T [L\mathcal{S}(f)]$$

subject to the constraint:

$$[\mathcal{S}(g) - D\mathcal{S}(f)]^T [\mathcal{S}(g) - D\mathcal{S}(f)] = \varepsilon,$$

where $\mathcal{S}$ is the stacking operator.

   (a) Prove that D is diagonalizable by $W = W_N \otimes W_N$, where $W_N(n,k) = \frac{1}{\sqrt{N}} e^{2\pi j \frac{nk}{N}}$, i.e. $W^{-1}DW$ is diagonal, and find its eigenvalues in terms of $DFT(h)$ where $D\mathcal{S}(\varphi) = \mathcal{S}(h * \varphi)$ for any $\varphi \in M_{N\times N}(\mathbb{R})$. You may consider using the results of Q2 and Q3(a) of Assignment 3. Also, you may refer to tutorial 7.

   (b) Given that the optimal solution f that solves the constrained least square problem satisfies $[\lambda D^T D + L^T L]\mathcal{S}(f) = \lambda D^T \mathcal{S}(g)$ for some parameter λ. Find $DFT(f)$ in terms of $DFT(g)$, $DFT(h)$, $DFT(p)$ and λ, where $L\mathcal{S}(\varphi) = \mathcal{S}(p * \varphi)$ for any $\varphi \in M_{N\times N}(\mathbb{R})$. You may consider using the result of Q3(b) of Assignment 3.

2. Given a noisy image $g(x, y)$, we consider the image denoising algorithm to obtain a clean image $f(x, y)$ through minimizing the following energy functional:

$$E(f) = \int_\Omega \{|f(x,y) - g(x,y) + \epsilon| + \lambda \|\nabla f(x,y)\|^2\}\, dx\, dy$$

where λ is a constant parameter, and ϵ is a small positive number.

   (a) Find the partial differential equation(s) that must be satisfied by the minimizer of $E(f)$.

   (b) In the discrete setting, design an iterative scheme to find a minimizer of the energy $E(f)$.

3. Given two observations of noisy image $g_1(x, y)$ and $g_2(x, y)$, we consider the image denoising algorithm to obtain a clean image $f(x, y)$ through minimizing the following energy functional:

$$E(f) = \int_\Omega \{|f(x,y) - g_1(x,y)|^2 + |f(x,y) - g_2(x,y)|^2 + \lambda \|\nabla f(x,y)\|^4\}\, dx\, dy$$

where λ is a constant parameter. Derive an iterative scheme to minimize $E(f)$ in the continuous setting.

4. Consider the following curve evolution model for image segmentation. Let $\gamma_t := \gamma_t(s) : [0, 2\pi] \to D$ be a closed curve in the image domain D. We proceed to find γ that minimizes:

$$E_{snake,2}(\gamma) = \int_0^{2\pi} \frac{1}{2}\|\gamma'(s)\|^2\, ds + \alpha \int_0^{2\pi} \frac{1}{2}\|\gamma''(s)\|^2\, ds + \beta \int_0^{2\pi} V(\gamma(s))\, ds,$$

where V is the edge detector, α and β are fixed positive parameters. Assume that γ' and γ'' are discretized by:

$$\gamma'(s_i) = [\gamma(s_{i+1}) - \gamma(s_i)]/\sigma \text{ and}$$
$$\gamma''(s_i) = [\gamma(s_{i+1}) - 2\gamma(s_i) + \gamma(s_{i-1})]/\sigma^2.$$

(a) Derive the gradient descent iterative scheme to minimize $E_{snake,2}$ in the continuous setting.

(b) Discretize $E_{snake,2}$.

(c) Derive the explicit Euler scheme (using gradient descent method) to iteratively minimize the discrete version of $E_{snake,2}$.