MATH3360: Mathematical Imaging

Assignment 4 Solutions

1. (a) D is block-circulant, i.e.

$$D = \begin{pmatrix} D_0 & D_{N-1} & D_{N-2} & \cdots & D_1 \\ D_1 & D_0 & D_{N-1} & \cdots & D_2 \\ D_2 & D_1 & D_0 & \cdots & D_{2,N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ D_{N-1} & D_{N-2} & D_{N-3} & \cdots & D_0 \end{pmatrix},$$

with each $N \times N$ block $D_0, D_1, \cdots, D_{N-1}$ being circulant. Hence denoting the (k,l)-th block of a $N \times N$ matrix A of $N \times N$ blocks by $A_{k,l}$, we have:

$$\begin{aligned}
(W^{-1}DW)_{k,l} &= \sum_{m=0}^{N-1} W^{-1}_{k,m}(DW)_{m,l} \\
&= \sum_{m=0}^{N-1} W^{-1}_{k,m} \sum_{n=0}^{N-1} D_{m,n} W_{n,l} \\
&= \sum_{m,n=0}^{N-1} \overline{W_N(k,m)} W_N(n,l) \overline{W_N} D_{m,n} W_N \\
&= \frac{1}{N} \sum_{m,n=0}^{N-1} e^{2\pi j \frac{ln-km}{N}} \overline{W_N} D_{m,n} W_N,
\end{aligned}$$

$$\begin{aligned}
\text{and thus } (W^{-1}DW)_{k,l}(q,r) &= \frac{1}{N} \sum_{m,n=0}^{N-1} e^{2\pi j \frac{ln-km}{N}} \sum_{s=0}^{N-1} \overline{W_N(q,s)} (D_{m,n}W_N)(s,r) \\
&= \frac{1}{N\sqrt{N}} \sum_{m,n,s=0}^{N-1} e^{2\pi j \frac{ln-km-qs}{N}} \sum_{t=0}^{N-1} D_{m,n}(s,t) W_N(t,r) \\
&= \frac{1}{N^2} \sum_{m,n,s,t=0}^{N-1} e^{2\pi j \frac{ln+rt-km-qs}{N}} D_{m-n,0}(s-t,0) \\
&= \frac{1}{N^2} \sum_{m,s=0}^{N-1} \sum_{n'=m-N+1}^{m} \sum_{t'=r-N+1}^{s} e^{2\pi j \frac{l(m-n')+r(s-t')-km-qs}{N}} D_{n',0}(t',0)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{N^2} \sum_{m,s=0}^{N-1} e^{2\pi j \frac{(l-k)m+(r-q)s}{N}} \sum_{n',t'=0}^{N-1} e^{2\pi j \frac{ln'+rt'}{N}} D(t'+n'N, 0) \\
&= \frac{1}{N^2} \cdot N\delta(l-k) \cdot N\delta(r-q) \cdot \sum_{n',t'=0}^{N-1} e^{-2\pi j \frac{ln'+rt'}{N}} h(t',n') \\
&= N^2 DFT(h)(r,l)\delta(l-k)\delta(r-q).
\end{aligned}$$

Hence $W^{-1}DW$ is diagonal.
Thus the eigenvalues of D are the diagonal entries of $W^{-1}DW$, i.e. $\{N^2 DFT(h)(u,v) : 0 \le u, v \le N-1\}$.

(b) Since L is also block-circulant, it is also diagonalizable by W. Denote $W^{-1}DW$ by Λ_D and $W^{-1}LW$ by Λ_L. Then

$$\begin{aligned}\lambda D^T D + L^T L &= \lambda D^* D + L^* L \\ &= \lambda (W\Lambda_D W^{-1})^*(W\Lambda_D W^{-1}) + (W\Lambda_L W^{-1})^*(W\Lambda_L W^{-1}) \\ &= \lambda W \Lambda_D^* \Lambda_D W^{-1} + W\Lambda_L^*\Lambda_L W^{-1}\end{aligned}$$

and $\lambda D^T = \lambda D^* = \lambda(W\Lambda_D W^{-1})^* = \lambda W \Lambda_D^* W^{-1}$. Hence

$$\begin{aligned}(\lambda W\Lambda_D^*\Lambda_D W^{-1} + W\Lambda_L^*\Lambda_L W^{-1})\mathcal{S}(f) &= \lambda W \Lambda_D^* W^{-1}\mathcal{S}(g), \\ (\lambda \Lambda_D^*\Lambda_D + \Lambda_L^*\Lambda_L)\mathcal{S}(DFT(f)) &= \lambda\Lambda_D^*\mathcal{S}(DFT(g))\end{aligned}$$

and thus

$$\begin{aligned}DFT(f)(u,v) &= \frac{\lambda N^2 \overline{DFT(h)(u,v)}}{N^4[\lambda|DFT(h)(u,v)|^2 + |DFT(p)(u,v)|^2]} DFT(g)(u,v) \\ &= \frac{1}{N^2}\frac{\lambda\overline{DFT(h)(u,v)}}{\lambda|DFT(h)(u,v)|^2 + |DFT(p)(u,v)|^2} DFT(g)(u,v).\end{aligned}$$

2. (a) $E(f) = \int_\Omega |f(x,y) - g(x,y) + \epsilon| + \lambda \big\|\nabla f(x,y)\big\|^2 \, dx\, dy$. Then for any $\varphi : \Omega \to \mathbb{R}$,

$$\begin{aligned}\frac{\partial E(f+t\varphi)}{\partial t}\Big|_{t=0} &= \frac{\partial}{\partial t}\Big|_{t=0}\int_\Omega |f + t\varphi - g + \epsilon| + \lambda\big\|\nabla(f+t\varphi)\big\|^2 \, dx\, dy \\ &= \int_\Omega \frac{1}{|f-g+\epsilon|}\varphi(f-g+\epsilon) + 2\lambda\nabla f\cdot\nabla\varphi \, dx\, dy \\ &= \int_\Omega \frac{1}{|f-g+\epsilon|}\varphi(f-g+\epsilon) - 2\lambda\varphi\Delta f \, dx\, dy + \int_{\partial\Omega} 2\lambda\varphi\nabla f\cdot\vec{n}\, d\sigma\end{aligned}$$

So, the minimizer of the energy must satisfy the following PDEs

$$\begin{cases}\dfrac{1}{\big|f(x,y)-g(x,y)+\epsilon\big|}(f(x,y)-g(x,y)+\epsilon) - 2\lambda\Delta f(x,y) = 0 & \text{if } (x,y)\in\Omega \\ \nabla f(x,y)\cdot\vec{n}(x,y) = 0 & \text{if } (x,y)\in\partial\Omega\end{cases}$$

(b) The iterative scheme in the continuous case is

$$f^{n+1}(x,y) = f^n(x,y) + t\bigg(- \frac{1}{\big|f^n(x,y)-g(x,y)+\epsilon\big|}(f^n(x,y)-g(x,y)+\epsilon) + 2\lambda\Delta f^n(x,y)\bigg)$$

for some prescribed small positive real number t. Numerically, we can approximate the Laplacian using the following formula

$$\Delta f^n(x,y)\, dt \approx f^n(x+1,y) + f^n(x-1,y) + f^n(x,y+1) + f^n(x,y-1) - 4f^n(x,y)$$

3. $E(f) = \int_\Omega |f(x,y)-g_1(x,y)|^2 + |f(x,y)-g_2(x,y)|^2 + \lambda\big\|\nabla f(x,y)\big\|^4 \, dx\, dy$. Then for any $\varphi : \Omega \to \mathbb{R}$,

$$\begin{aligned}
\frac{\partial E(f+t\varphi)}{\partial t}\Big|_{t=0} &= \frac{\partial}{\partial t}\Big|_{t=0} \int_\Omega |f+t\varphi-g_1|^2 + |f+t\varphi-g_2|^2 + \lambda\big\|\nabla(f+t\varphi)\big\|^4 \, dx\, dy \\
&= \frac{\partial}{\partial t}\Big|_{t=0} \int_\Omega |f+t\varphi-g_1|^2 + |f+t\varphi-g_2|^2 + \lambda\Big(\big\|\nabla(f+t\varphi)\big\|^2\Big)^2 \, dx\, dy \\
&= \int_\Omega 2\varphi\,(f-g_1) + 2\varphi\,(f-g_2) + 4\lambda\|\nabla f\|^2\,\nabla f\cdot\nabla\varphi \, dx\, dy \\
&= 2\int_\Omega \varphi(2f-g_1-g_2) - 2\lambda\varphi\,\nabla\cdot\big(\|\nabla f\|^2\,\nabla f\big)\, dx\, dy + \int_{\partial\Omega} 2\lambda\varphi\|\nabla f\|^2\,\nabla f\cdot\vec{n}\, d\sigma
\end{aligned}$$

Ignoring the boundary condition, the iterative scheme in the continuous case is

$$f^{n+1}(x,y) = f^n(x,y) + t\Big(g_1(x,y) + g_2(x,y) - 2f(x,y) + 2\lambda\nabla\cdot\big(\|\nabla f\|^2\,\nabla f\big)(x,y)\Big)$$

where t is a small prescribed positive real number.

4. (a) For any $\varphi : [0,2\pi] \to D$,

$$\begin{aligned}
&\frac{\partial E_{snake,2}(\gamma+t\varphi)}{\partial t}\Big|_{t=0} \\
&= \frac{\partial}{\partial t}\Big|_{t=0}\left\{\frac{1}{2}\int_0^{2\pi}\|(\gamma+t\varphi)'(s)\|^2\,ds + \frac{\alpha}{2}\int_0^{2\pi}\|(\gamma+t\varphi)''(s)\|^2\,ds + \beta\int_0^{2\pi}V((\gamma+t\varphi)(s))\,ds\right\} \\
&= \int_0^{2\pi}\gamma'(s)\varphi'(s)\,ds + \alpha\int_0^{2\pi}\gamma''(s)\varphi''(s)\,ds + \beta\int_0^{2\pi}\nabla V(\gamma(s))\varphi(s)\,ds \\
&= \gamma'(s)\varphi(s)\Big|_{s=0}^{2\pi} - \int_0^{2\pi}\gamma''(s)\varphi(s)\,ds + \alpha\gamma''(s)\varphi'(s)\Big|_{s=0}^{2\pi} - \alpha\int_0^{2\pi}\gamma'''(s)\varphi(s)\,ds \\
&\quad + \beta\int_0^{2\pi}\nabla V(\gamma(s))\varphi(s)\,ds \\
&= \int_0^{2\pi}\varphi(s)[\alpha\gamma^{(4)}(s) + \beta\nabla V(\gamma(s)) - \gamma''(s)]\,ds.
\end{aligned}$$

Hence a descent direction is given by:

$$\varphi(s) = -\alpha\gamma^{(4)}(s) - \beta\nabla V(\gamma(s)) + \gamma''(s),$$

which yields the gradient descent equation:

$$\frac{\partial\gamma(s;t)}{\partial t} = -\alpha\gamma^{(4)}(s) - \beta\nabla V(\gamma(s)) + \gamma''(s)$$

and the corresponding iterative scheme:

$$\gamma^{n+1}(s) = \gamma^n(s) + \Delta t[-\alpha\gamma^{n(4)}(s) - \beta\nabla V(\gamma^n(s)) + \gamma^{n\prime\prime}(s)],$$

where Δt is the time step.

Remark. *Note that since D is typically two-dimensional, γ and φ are vector-valued functions. Hence the integrations by parts are written in abuse of notation. If the reader is troubled, it is advised that they consider the integrals on each coordinate separately.*

(b)

$$E_{\substack{snake,2\\ discrete}}(\gamma) = \frac{1}{2}\sum_{i=1}^{N}\frac{[\gamma(s_{i+1})-\gamma(s_i)]^2}{\sigma^2} + \frac{\alpha}{2}\sum_{i=1}^{N}\frac{[\gamma(s_{i+1})-2\gamma(s_i)+\gamma(s_{i-1})]^2}{\sigma^4} + \beta\sum_{i=1}^{N}V(\gamma(s_i)).$$

(c)

$$
\begin{aligned}
&\frac{\partial \underset{discrete}{E_{snake,2}}(\gamma)}{\partial \gamma(s_i)} \\
&= \frac{\partial}{\partial \gamma(s_i)} \left\{ \frac{1}{2} \sum_{j=1}^{N} \frac{[\gamma(s_{j+1}) - \gamma(s_j)]^2}{\sigma^2} + \frac{\alpha}{2} \sum_{j=1}^{N} \frac{[\gamma(s_{j+1}) - 2\gamma(s_j) + \gamma(s_{j-1})]^2}{\sigma^4} + \beta \sum_{j=1}^{N} V(\gamma(s_j)) \right\} \\
&= \frac{-\gamma(s_{i-1}) + 2\gamma(s_i) - \gamma(s_{i+1})}{\sigma^2} + \frac{\alpha[\gamma(s_{j-2}) - 4\gamma(s_{j-1}) + 6\gamma(s_j) - 4\gamma(s_{j+1}) + \gamma(s_{j+2})]}{\sigma^4} \\
&\quad + \beta \nabla V(\gamma(s_i)).
\end{aligned}
$$

Hence guided by the gradient descent equation:

$$
\frac{\partial \gamma}{\partial t} = -\nabla \underset{discrete}{E_{snake,2}}(\gamma),
$$

we iteratively minimize $\underset{discrete}{E_{snake,2}}(\gamma)$ by updating γ:

$$
\begin{aligned}
\frac{\gamma^{n+1}(s_i) - \gamma^n(s_i)}{\Delta t} &= \frac{\gamma^n(s_{i-1}) - 2\gamma^n(s_i) + \gamma^n(s_{i+1})}{\sigma^2} \\
&+ \frac{\alpha[-\gamma^n(s_{j-2}) + 4\gamma^n(s_{j-1}) - 6\gamma^n(s_j) + 4\gamma^n(s_{j+1}) - \gamma^n(s_{j+2})}{\sigma^4} \\
&- \beta \nabla V(\gamma^n(s_i)),
\end{aligned}
$$

where Δt is the time step.