MATH3360: Mathematical Imaging

Assignment 2

Due: 19 October 2021

1. Let $\mathcal{H} := \{H_m : m \in \mathbb{N} \cup \{0\}\}$ be the sequence of Haar functions. Consider the inner product space $(L^2(\mathbb{R}), \langle \cdot, \cdot \rangle)$ where

$$L^2(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{R} \,\middle|\, \int_{\mathbb{R}} f^2 < \infty \right\},$$

and for any $f, g \in L^2(\mathbb{R})$,

$$\langle f, g \rangle = \int_{\mathbb{R}} fg.$$

   (a) (Unit) Prove that $\int_{\mathbb{R}} [H_m(t)]^2 \, dt = 1$ for any $m \in \mathbb{N} \cup \{0\}$. (Hence $H_m \in L^2(\mathbb{R})$ and $\|H_m\| = 1$.)

   (b) (Orthogonality)

      i. Prove that $\langle H_0, H_m \rangle = 0$ for any $m \in \mathbb{N} \setminus \{0\}$.

      ii. Let $m_1, m_2 \in \mathbb{N}$ such that $0 \neq m_1 < m_2$. Then $m_1 = 2^{p_1} + n_1$ and $m_2 = 2^{p_2} + n_2$ for some $p_1, p_2 \in \mathbb{N} \cup \{0\}$, $n_1 \in \mathbb{Z} \cap [0, 2^{p_1} - 1]$ and $n_2 \in \mathbb{Z} \cap [0, 2^{p_2} - 1]$.

         A. Suppose $p_1 = p_2$. Prove that $\langle H_{m_1}, H_{m_2} \rangle = 0$.
         Hint. In this case $n_1 < n_2$.

         B. Suppose $p_1 < p_2$. Prove that $\langle H_{m_1}, H_{m_2} \rangle = 0$.
         Hint. Consider the possible subset relations between the supports of H_{m_1} and H_{m_2}.

   The above establishes that $\mathcal{H}$ is orthonormal in $(L^2(\mathbb{R}), \langle \cdot, \cdot, \rangle)$.

2. Let $\mathcal{W} := \{W_m : m \in \mathbb{N} \cup \{0\}\}$ be the sequence of Walsh functions. Consider $(L^2(\mathbb{R}), \langle \cdot, \cdot \rangle)$ defined in Q1.

   (a) (Unit) Prove that $\int_{\mathbb{R}} [W_m(t)]^2 \, dt = 1$ for any $m \in \mathbb{N} \cup \{0\}$. (Hence $W_m \in L^2(\mathbb{R})$ and $\|W_m\| = 1$.)

   (b) (Orthogonality) Let $P(m)$ be the proposition that $\{W_0, \ldots, W_m\}$ is orthogonal in $(L^2(\mathbb{R}), \langle \cdot, \cdot \rangle)$. $P(0)$ is vacuously true. We aim to establish:

$$P(m) \text{ is true} \implies P(2m+1) \text{ is true} \qquad (\star)$$

whenever $m \in \mathbb{N} \cup \{0\}$. Also note that $P(m)$ implies $P(m-1)$ for any $m \in \mathbb{N} \setminus \{0\}$ (since the latter concerns the orthogonality of a subset of the set concerned by the former), so by induction we establish $P(m)$ to be true for all $m \in \mathbb{N} \setminus \{0\}$ once $(\star)$ holds.

Suppose $P(k)$ holds for some $k \in \mathbb{N} \cup \{0\}$. Let $m_1, m_2 \in \mathbb{Z} \cap [0, 2k+1]$ such that $m_1 < m_2$. Then $m_1 = 2j_1 + q_1$ and $m_2 = 2j_2 + q_2$ for some $j_1, j_2 \in \mathbb{N} \cup \{0\}$ and $q_1, q_2 \in \{0, 1\}$.

i. Suppose $j_1 = j_2$. Prove that $\langle W_{m_1}, W_{m_2} \rangle = 0$.
Hint. In this case $q_1 = 0$, $q_2 = 1$.

ii. Suppose $j_1 < j_2$. Prove that $\langle W_{m_1}, W_{m_2} \rangle = 0$.
Hint. $j_2 \leq n$. Make good use of the induction hypothesis.

The above establishes that $\mathcal{W}$ is orthonormal in $(L^2(\mathbb{R}), \langle \cdot, \cdot \rangle)$.

3. Let $A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

   (a) Compute the DFT of A and B.

   (b) Discard 4 smallest (in absolute value) non-zero coefficients in the DFT of A, then write the reconstructed A. (Truncate all the complex numbers in the final answer if any)

   (c) Compute $A * B$ and DFT of $A * B$. Check whether $\widehat{A * B}(p, q) = 16\widehat{A}(p, q)\widehat{B}(p, q)$ for all p, q.

4. In this problem, please ignore the definition of the DFT in the lecture notes. Consider this alternative definition for the DFT for $N \times N$ images:

$$\hat{f}(m, n) = DFT(f)(m, n) = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} f(k, l) e^{2\pi j \frac{mk+nl}{N}}.$$

   (a) Show that the inverse DFT (iDFT) is defined by

$$f(p, q) = iDFT(\hat{f})(p, q) = \frac{1}{N} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \hat{f}(m, n) e^{-2\pi j \frac{pm+qn}{N}}.$$

   (b) Determine the matrix U used to calculate the DFT of an $N \times N$ image, i.e. $\hat{f} = U f U$ and show that U is unitary

   (c) In the lecture notes, we have a formula for DFT of convolution. Derive a similar formula for it under this new definition.

(d) Derive a similar formula for DFT of shifted images under this new definition.

5. **Coding assignment**: Please read the MATLAB file in the attached zip file carefully. There are missing lines in the file. Add the missing lines by yourself and test the file using the given image. (Note: In this coding assignment, we discuss the image processing of grayscale images only.)

   Coding instruction:

   Q1: Recall that DFT can be rewritten as matrix multiplication.

$$\hat{g} = UgU \tag{1}$$

   where $U_{\alpha\beta} = \frac{1}{N} e^{-2\pi j \frac{\alpha\beta}{N}}$ where $0 \leq \alpha, \beta \leq N-1$, and $U = (U_{\alpha\beta})_{0 \leq \alpha, \beta \leq N-1} \in M_{N\times N}(\mathbb{C})$.

   In this coding assignment, you are required to reconstruct the image given a modified $\hat{g}$, which represents the Fourier coefficients. You are not allowed to use the built-in MATLAB function $ifft2$.

   Later in this course, we will do image processing in the spectral domain. We will use this technique again and again.