MATH3360: Mathematical Imaging

Assignment 1 Solutions

1. (a) Note that H is a 4×4 matrix; hence it represents a linear transformation on 2×2 images.

   H is not block-circulant. For example, consider the $y = 1, \beta = 1$-submatrix of H, i.e. $\begin{pmatrix} 2 & 0 \\ 4 & 2 \end{pmatrix}$. This is not a circulant matrix, as the shift-operator T maps $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ to $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ instead of $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$. Hence h is not shift-invariant with h_s being 2-periodic in both arguments.
   (However, H is block-Toeplitz and thus h is shift-invariant.)

   H is not a Kronecker product of two 2×2 matrices. For example, consider the $y = 1, \beta = 1$- and $y = 2, \beta = 1$-submatrices of H, i.e. $\begin{pmatrix} 2 & 0 \\ 4 & 2 \end{pmatrix}$ and $\begin{pmatrix} 2 & 2 \\ 0 & 4 \end{pmatrix}$. Neither is a scalar multiple of the other. Hence h is not separable.

   (b) Note that H is a 9×9 matrix; hence it represents a linear transformation on 3×3 images.

   H is block-circulant. The $y = 1, \beta = 1$-, the $y = 2, \beta = 2$- and the $y = 3, \beta = 3$-submatrices of H are all $\begin{pmatrix} 9 & 9 & 18 \\ 18 & 9 & 9 \\ 9 & 18 & 9 \end{pmatrix}$, which is circulant; the $y = 2, \beta = 1$-, the $y = 3, \beta = 2$- and the $y = 1, \beta = 3$-submatrices of H are all $\begin{pmatrix} 9 & 9 & 18 \\ 18 & 9 & 9 \\ 9 & 18 & 9 \end{pmatrix}$, which is circulant; the $y = 3, \beta = 1$-, the $y = 1, \beta = 2$- and the $y = 2, \beta = 3$-submatrices of H are all $\begin{pmatrix} 18 & 18 & 36 \\ 36 & 18 & 18 \\ 18 & 36 & 18 \end{pmatrix}$, which is also circulant. Hence h is shift-invariant with h_s being 3-periodic in both arguments.

   H is the Kronecker product of two 3×3 matrices; explicitly,

   $$H = \begin{pmatrix} 3 & 3 & 6 \\ 6 & 3 & 3 \\ 3 & 6 & 3 \end{pmatrix} \otimes \begin{pmatrix} 3 & 3 & 6 \\ 6 & 3 & 3 \\ 3 & 6 & 3 \end{pmatrix} = \left[\operatorname{circ}\left((3,3,6)^T\right)\right]^T \otimes \left[\operatorname{circ}\left((3,3,6)^T\right)\right]^T.$$

   Hence h is separable.

   (c) Let $s = \alpha - x, t = \beta - y$. Then, $H(x, \alpha, y, \beta) = st + s^2$. Hence, H is shift-invariant.

Suppose H is separable. Then, there exists h_c, h_r such that $H(x, \alpha, y, \beta) = h_c(x, \alpha)h_r(y, \beta)$. We can then deduce the following results:
$H(1, 2, 2, 3) = h_c(1.2)h_r(2, 3) = 2$ and $H(1, 2, 2, 4) = h_c(1, 2)h_r(2, 3) = 3$
$\implies h_r(2, 3)/h_r(2, 4) = 2/3$.
But, $H(3, 2, 2, 3) = h_c(1, 2)h_r(2, 3) = 0$ and $H(3, 2, 2, 4) = h_c(1, 2)h_r(2, 4) = -1$
$\implies h_r(2, 3)/h_r(2, 4) = 0 \neq 2/3$.
Hence, H is not separable.

(d) Since $H(x, \alpha, y, \beta) = \alpha\beta e^{(x-y)(x^2+xy+y^2)} = \alpha\beta e^{x^3-y^3} = \alpha e^{x^3}\beta e^{-y^3}$, H is separable.
Note that $H(1, 2, 1, 1) = 2$, but $H(2, 3, 1, 1) = 3e^7 \neq 2$. Hence, H is not shift-invariant.

2. Let $f, g \in M_{m\times n}(\mathbb{R})$, and assume that they are periodically extended.

   Let $\alpha \in \mathbb{N} \cap [1, m]$ and $\beta \in \mathbb{N} \cap [1, n]$. By definition,

$$
\begin{aligned}
f * g(\alpha, \beta) &= \sum_{x=1}^{m}\sum_{y=1}^{n} f(x, y)g(\alpha - x, \beta - y) \\
&= \sum_{i=\alpha-m}^{\alpha-1}\sum_{j=\beta-n}^{\beta-1} f(\alpha - i, \beta - j)g(i, j) \text{ (letting } i = \alpha - x,\ j = \beta - y) \\
&= \sum_{i=\alpha-m}^{0}\sum_{j=\beta-n}^{0} f(\alpha - i, \beta - j)g(i, j) + \sum_{i=\alpha-m}^{0}\sum_{j=1}^{\beta-1} f(\alpha - i, \beta - j)g(i, j) \\
&\quad + \sum_{i=1}^{\alpha-1}\sum_{j=\beta-n}^{0} f(\alpha - i, \beta - j)g(i, j) + \sum_{i=1}^{\alpha-1}\sum_{j=1}^{\beta-1} f(\alpha - i, \beta - j)g(i, j) \\
&= \sum_{i=\alpha}^{m}\sum_{j=\beta}^{n} f(\alpha - i, \beta - j)g(i, j) + \sum_{i=\alpha}^{m}\sum_{j=1}^{\beta-1} f(\alpha - i, \beta - j)g(i, j) \\
&\quad + \sum_{i=1}^{\alpha-1}\sum_{j=\beta}^{n} f(\alpha - i, \beta - j)g(i, j) + \sum_{i=1}^{\alpha-1}\sum_{j=1}^{\beta-1} f(\alpha - i, \beta - j)g(i, j) \text{ (by periodicity)} \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n} g(i, j)f(\alpha - i, \beta - j) \\
&= g * f(\alpha, \beta);
\end{aligned}
$$

   hence $f * g = g * f$.

3. Let h be the shift-invariant PSF of a linear image transformation on $M_{n\times n}(\mathbb{R})$ in the sense that $h(x, \alpha, y, \beta) = h_s(\alpha - x, \beta - y)$. Let H be the corresponding transformation matrix.

Fix y and β. Then for any α, x and $a \in \mathbb{N}$ satisfying $a \leq n - \max\{\alpha, x\}$,

$$\begin{aligned} h\left(x+an, \alpha+an, y, \beta\right) &= h_s\left(\alpha+an-x-an, \beta-y\right) \\ &= h_s(\alpha-x, \beta-y) \\ &= h(x, \alpha, y, \beta) \end{aligned}$$

On the other hand, fix x and α. Then for any β, y and $a \in \mathbb{N}$ satisfying $a \leq n - \max\{\beta, y\}$,

$$\begin{aligned} h(\alpha, x, \beta+an, y+an) &= h_s(\alpha-x, \beta+an-y-an) \\ &= h_s(\alpha-x, \beta-y) \\ &= h(x, \alpha, y, \beta) \end{aligned}$$

Hence, we know H is block Toeplitz.
Reverse all the statements shown above, we know h is shift-invariant if H is block Toeplitz.

4. Let h be the separable PSF of a linear image transformation, with $h(x, \alpha, y, \beta) = h_c(x, \alpha)h_r(y, \beta)$. Let H be the corresponding transformation matrix.

   Then the $y = k, \beta = l$-submatrix of H (denoted by $\tilde{H}_{kl}$) is given by

$$\begin{aligned} \left(\begin{array}{cc} & x \rightarrow \\ \alpha \downarrow & \begin{pmatrix} y = k \\ \beta = l \end{pmatrix} \end{array} \right) &= \left[H\left(\alpha + (l-1)n, x + (k-1)n\right)\right]_{\substack{1 \leq x \leq n \\ 1 \leq \alpha \leq n}} \\ &= [h(x, \alpha, k, l)]_{\substack{1 \leq x \leq n \\ 1 \leq \alpha \leq n}} \\ &= [h_c(x, \alpha)h_r(k, l)]_{\substack{1 \leq x \leq n \\ 1 \leq \alpha \leq n}} \\ &= h_r(k, l)[h_c(x, \alpha)]_{\substack{1 \leq x \leq n \\ 1 \leq \alpha \leq n}} \\ &= h_r(k, l)h_c^T. \end{aligned}$$

Recall that

$$H = \begin{pmatrix} \begin{pmatrix} & x \rightarrow \\ \alpha \downarrow & \begin{pmatrix} y=1 \\ \beta=1 \end{pmatrix} \end{pmatrix} & \begin{pmatrix} & x \rightarrow \\ \alpha \downarrow & \begin{pmatrix} y=2 \\ \beta=1 \end{pmatrix} \end{pmatrix} & \cdots & \begin{pmatrix} & x \rightarrow \\ \alpha \downarrow & \begin{pmatrix} y=n \\ \beta=1 \end{pmatrix} \end{pmatrix} \\ \begin{pmatrix} & x \rightarrow \\ \alpha \downarrow & \begin{pmatrix} y=1 \\ \beta=2 \end{pmatrix} \end{pmatrix} & \begin{pmatrix} & x \rightarrow \\ \alpha \downarrow & \begin{pmatrix} y=2 \\ \beta=2 \end{pmatrix} \end{pmatrix} & \cdots & \begin{pmatrix} & x \rightarrow \\ \alpha \downarrow & \begin{pmatrix} y=n \\ \beta=2 \end{pmatrix} \end{pmatrix} \\ \vdots & \vdots & \ddots & \vdots \\ \begin{pmatrix} & x \rightarrow \\ \alpha \downarrow & \begin{pmatrix} y=1 \\ \beta=n \end{pmatrix} \end{pmatrix} & \begin{pmatrix} & x \rightarrow \\ \alpha \downarrow & \begin{pmatrix} y=2 \\ \beta=n \end{pmatrix} \end{pmatrix} & \cdots & \begin{pmatrix} & x \rightarrow \\ \alpha \downarrow & \begin{pmatrix} y=n \\ \beta=n \end{pmatrix} \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} \tilde{H}_{11} & \tilde{H}_{21} & \cdots & \tilde{H}_{n1} \\ \tilde{H}_{12} & \tilde{H}_{22} & \cdots & \tilde{H}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{H}_{1n} & \tilde{H}_{2n} & \cdots & \tilde{H}_{nn} \end{pmatrix} = \begin{pmatrix} h_r(1,1)h_c^T & h_r(2,1)h_c^T & \cdots & h_r(n,1)h_c^T \\ h_r(1,2)h_c^T & h_r(2,2)h_c^T & \cdots & h_r(n,2)h_c^T \\ \vdots & \vdots & \ddots & \vdots \\ h_r(1,n)h_c^T & h_r(2,n)h_c^T & \cdots & h_r(n,n)h_c^T \end{pmatrix}$$

$$= \begin{pmatrix} h_r^T(1,1)h_c^T & h_r^T(1,2)h_c^T & \cdots & h_r^T(1,n)h_c^T \\ h_r^T(2,1)h_c^T & h_r^T(2,2)h_c^T & \cdots & h_r^T(2,n)h_c^T \\ \vdots & \vdots & \ddots & \vdots \\ h_r^T(n,1)h_c^T & h_r^T(n,2)h_c^T & \cdots & h_r^T(n,n)h_c^T \end{pmatrix} = h_r^T \otimes h_c^T.$$

5. (a) We first compute the SVD decomposition of A. We start by finding the eigenvalues and corresponding orthonormal eigenbasis of A^TA.

$$A^TA = \begin{pmatrix} 10 & 6 & 0 \\ 6 & 10 & 0 \\ 0 & 0 & 25 \end{pmatrix}$$

$p(\lambda) = (det)(A^TA - \lambda I_3) = -(\lambda-4)(\lambda-16)(\lambda-25)$.
So, the eigenvalues of A^TA are $\lambda_1 = 25, \lambda_2 = 16, \lambda_3 = 4$.

The corresponding eigenvectors are $v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, and $v_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$.

Then we compute the matrix U. $u_1 = \frac{1}{\sigma_1} Av_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $u_2 = \frac{1}{\sigma_2} Av_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, and $u_3 = \frac{1}{\sigma_3} Av_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

So, $A = U\Sigma V^T = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{pmatrix}$

We can then write A as $A = 5u_1v_1^T + 4u_2v_2^T + 2u_3v_3^T = 5\begin{pmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 1\end{pmatrix} + 2\begin{pmatrix}1 & 1 & 0\\1 & 1 & 0\\0 & 0 & 0\end{pmatrix} + \begin{pmatrix}1 & -1 & 0\\-1 & 1 & 0\\0 & 0 & 0\end{pmatrix}$

(b) By the formula for error of rank-k approximation of SVD, the rank-2 approximation of A has error $\sigma_3 = 2$ in Frobenius norm. So, we can simply take

$$\tilde{A} = 5u_1v_1^T + 4u_2v_2^T = 5\begin{pmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 1\end{pmatrix} + 2\begin{pmatrix}1 & 1 & 0\\1 & 1 & 0\\0 & 0 & 0\end{pmatrix}$$

6. **Coding Assignment:**

Q1:

```
    ijv(ind, :) = [(beta-1)*w+alpha, (y-1)*w+x, kernel(i+2, ...
        j+2)];
```

Q2:

```
    hr_1 = eye(H);
    hr_2 = circshift(hr_1, [1 0]);
    hr_3 = circshift(hr_1, [-1 0]);
    hr = (hr_1*2 + hr_2 + hr_3) / 4;
```

Q3:

```
    img = img + S(i, i) * U(:, i) * V(:, i)';
```

Q4:

```
    if t ≥ n/(2^p) && t < (n+0.5)/(2^p)
        H(i, j) = sqrt(2)^p;
    elseif t ≥ (n+0.5)/(2^p) && t < (n+1)/(2^p)
        H(i, j) = -sqrt(2)^p;
```

```
    img = img + G(i, j) * Ht(:, i) * H(j, :);
```