MATH1010 University Mathematics
Limits of sequences

1. Let $a_n = \frac{1}{n^2+1} + \frac{1}{n^2+2} + \frac{1}{n^2+3} + \cdots + \frac{1}{n^2+n}$. Find $\lim_{n\to\infty} a_n$.

2. Let $a_n = \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n-1}$. Prove that a_n is convergent and $\lim_{n\to\infty} a_n > 0$.

3. Prove that $k(n-k+1) \geq n$ for any $1 \leq k \leq n$. Hence prove that $\lim_{n\to\infty} \frac{1}{\sqrt[n]{n!}} = 0$.

Solution:

1. For any $n = 1, 2, 3, \ldots$, we have

$$\begin{aligned} 0 &< \frac{1}{n^2+1} + \frac{1}{n^2+2} + \frac{1}{n^2+3} + \cdots + \frac{1}{n^2+n} \\ &\leq \frac{1}{n^2} + \frac{1}{n^2} + \frac{1}{n^2} + \cdots + \frac{1}{n^2} \\ &= \frac{1}{n}. \end{aligned}$$

Now

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

By squeeze theorem, we have

$$\lim_{n\to\infty} \left(\frac{1}{n^2+1} + \frac{1}{n^2+2} + \frac{1}{n^2+3} + \cdots + \frac{1}{n^2+n} \right) = 0.$$

2. Observe that

$$\begin{aligned} & a_{n+1} - a_n \\ = & \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n+1} \right) - \left(\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n-1} \right) \\ = & \frac{1}{2n} + \frac{1}{2n+1} - \frac{1}{n} \\ < & \frac{1}{2n} + \frac{1}{2n} - \frac{1}{n} \\ = & 0. \end{aligned}$$

Thus a_n is strictly decreasing. Now

$$a_n = \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n-1} > \frac{1}{2n} + \frac{1}{2n} + \cdots + \frac{1}{2n} = \frac{1}{2}$$

It follows by monotone convergence theorem that a_n is convergent. Moreover we have $\lim_{n\to\infty} a_n \geq \frac{1}{2} > 0$.
Remark: It can be proved that $\lim_{n\to\infty} a_n = \ln 2 \approx 0.6931$.

3. For any $1 \leq k \leq n$, we have

$$\begin{aligned} k(n-k+1) - n &= kn - k^2 + k - n \\ &= kn - n - k^2 + k \\ &= n(k-1) - k(k-1) \\ &= (n-k)(k-1) \\ &\geq 0. \end{aligned}$$

Hence for any positive integer n,

$$\begin{aligned} \frac{1}{\sqrt[n]{n!}} &= \frac{1}{(n!)^{\frac{1}{n}}} \\ &= \frac{1}{((1\cdot n)\times(2\cdot(n-1))\times(3\cdot(n-2))\times\cdots\times(n\cdot 1))^{\frac{1}{2n}}} \\ &\leq \frac{1}{(n\times n\times n\times\cdots\times n)^{\frac{1}{2n}}} \\ &= \frac{1}{(n^n)^{\frac{1}{2n}}} \\ &= \frac{1}{\sqrt{n}}. \end{aligned}$$

Thus

$$0 < \frac{1}{\sqrt[n]{n!}} \leq \frac{1}{\sqrt{n}}.$$

Now

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0.$$

It follows by squeeze theorem that

$$\lim_{n\to\infty} \frac{1}{\sqrt[n]{n!}} = 0.$$