MATH1010 University Mathematics
Fundamental theorem of calculus

1. Evaluate the following definite integrals using Riemann sum.

   (a) $\int_0^1 x^3 dx$ (b) $\int_0^1 e^{2x} dx$ (c) $\int_0^{\frac{\pi}{2}} \sin^2 x dx$

2. Use a suitable integral to evaluate the following limits.

   (a) $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{(n+k)^2} = \lim_{n\to\infty} \left(\frac{n}{(n+1)^2} + \frac{n}{(n+2)^2} + \cdots + \frac{n}{(2n)^2} \right)$

   (b) $\lim_{n\to\infty} \frac{1}{n^{\frac{3}{2}}} \sum_{k=1}^{n} \sqrt{k} = \lim_{n\to\infty} \frac{1}{n^{\frac{3}{2}}} (1 + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{n})$

   (c) $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \sin^2 \left(\frac{k\pi}{12n} \right) = \lim_{n\to\infty} \frac{1}{n} \left(\sin^2 \left(\frac{\pi}{12n} \right) + \sin^2 \left(\frac{2\pi}{12n} \right) + \cdots + \sin^2 \left(\frac{n\pi}{12n} \right) \right)$

   (d) $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{4n^2 - k^2}} = \lim_{n\to\infty} \left(\frac{1}{\sqrt{4n^2 - 1^2}} + \frac{1}{\sqrt{4n^2 - 2^2}} + \cdots + \frac{1}{\sqrt{4n^2 - n^2}} \right)$

3. Evaluate the following definite integrals.

   (a) $\int_{-1}^{3} \frac{x}{1+x^2} dx$

   (b) $\int_0^{\frac{\pi}{2}} \sin^3 x \cos^2 x \, dx$

   (c) $\int_0^{\frac{\pi}{6}} x^2 \cos x \, dx$

   (d) $\int_0^1 e^{\sqrt{x}} dx$

   (e) $\int_{-2}^{5} |x^2 - 4| dx$

   (f) $\int_{-1}^{2} e^{|x-1|} dx$

4. Evaluate

   (a) $\frac{d}{dx} \int_0^x e^{t^2} dt$

   (b) $\frac{d}{dx} \int_1^{x^2} \sqrt{1 + e^t} \, dt$

   (c) $\frac{d}{dx} \int_x^{\sqrt{x}} \ln(1 + t^2) dt$

   (d) $\frac{d}{dx} \int_{-\sqrt{x}}^{x} \sin(t^2) dt$

   (e) $\frac{d}{dx} \int_0^x x \sin(x^2 t^2) dt$

   (f) $\frac{d}{dx} \int_1^x \frac{e^{xt} dt}{t}$

5. Evaluate the following limits.

(a) $\lim_{x\to 0} \frac{\int_0^x t\ln(1+t^2)dt}{x^4}$

(b) $\lim_{x\to 0} \int_0^x \frac{e^{t^2}-1}{x^3}\,dt$

(c) $\lim_{x\to 0} \frac{\int_0^x \sin 6x \sin(t^2)dt}{x^4}$

(d) $\lim_{x\to 0} \frac{\int_0^x (e^{x^2t^2}-1)dt}{x^5}$

Solution:

1. (a)

$$\begin{aligned}
\int_0^1 x^3 dx &= \lim_{n\to\infty} \frac{1}{n}\left(\frac{1^3}{n^3}+\frac{2^3}{n^3}+\frac{3^3}{n^3}+\cdots+\frac{n^3}{n^3}\right) \\
&= \lim_{n\to\infty} \frac{n^2(n+1)^2}{4n^4} \\
&= \frac{1}{4}
\end{aligned}$$

(b)

$$\begin{aligned}
\int_0^1 e^{2x} dx &= \lim_{n\to\infty} \frac{1}{n}\left(e^{\frac{2}{n}}+e^{\frac{4}{n}}+e^{\frac{6}{n}}+\cdots+e^{\frac{2n}{n}}\right) \\
&= \lim_{n\to\infty} \frac{e^{\frac{2}{n}}((e^{\frac{2}{n}})^n-1)}{n(e^{\frac{2}{n}}-1)} \\
&= \lim_{n\to\infty} \frac{e^{\frac{2}{n}}(e^2-1)}{n(e^{\frac{2}{n}}-1)} \\
&= (e^2-1)\lim_{n\to\infty} \frac{\frac{1}{n}}{e^{\frac{2}{n}}-1} \\
&= (e^2-1)\lim_{y\to 0} \frac{y}{e^{2y}-1} \\
&= \frac{e^2-1}{2}
\end{aligned}$$

(c)

$$\begin{aligned}
& \int_0^{\frac{\pi}{2}} \sin^2 x dx \\
= & \lim_{n\to\infty} \frac{\frac{\pi}{2}-0}{n}\left(\sin^2\left(\frac{\pi}{2n}\right)+\sin^2\left(\frac{2\pi}{2n}\right)+\cdots+\sin^2\left(\frac{n\pi}{2n}\right)\right) \\
= & \lim_{n\to\infty} \frac{\pi}{4n}\left(\left(\sin^2\left(\frac{0\pi}{2n}\right)+\sin^2\left(\frac{n\pi}{2n}\right)\right)+\left(\sin^2\left(\frac{\pi}{2n}\right)+\sin^2\left(\frac{(n-1)\pi}{2n}\right)\right)+\cdots\right. \\
& \left.\cdots+\left(\sin^2\left(\frac{(n-1)\pi}{2n}\right)+\sin^2\left(\frac{\pi}{2n}\right)\right)+\left(\sin^2\left(\frac{n\pi}{2n}\right)+\sin^2\left(\frac{0\pi}{2n}\right)\right)\right) \\
= & \lim_{n\to\infty} \frac{\pi}{4n}\left(\left(\sin^2\left(\frac{0\pi}{2n}\right)+\cos^2\left(\frac{0\pi}{2n}\right)\right)+\left(\sin^2\left(\frac{\pi}{2n}\right)+\cos^2\left(\frac{\pi}{2n}\right)\right)+\right. \\
& \left.\cdots+\left(\sin^2\left(\frac{(n-1)\pi}{2n}\right)+\cos^2\left(\frac{(n-1)\pi}{2n}\right)\right)+\left(\sin^2\left(\frac{n\pi}{2n}\right)+\cos^2\left(\frac{n\pi}{2n}\right)\right)\right) \\
= & \lim_{n\to\infty} \frac{n\pi}{4n} \\
= & \frac{\pi}{4}
\end{aligned}$$

2. (a)

$$\begin{aligned}
\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{(n+k)^2} &= \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{(1+\frac{k}{n})^2} \\
&= \int_0^1 \frac{dx}{(1+x)^2} \\
&= \left[-\frac{1}{1+x}\right]_0^1 \\
&= \frac{1}{2}
\end{aligned}$$

(b)

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n^{\frac{3}{2}}} \sum_{k=1}^{n} \sqrt{k} &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \sqrt{\frac{k}{n}} \\
&= \int_0^1 \sqrt{x}\, dx \\
&= \left[\frac{2x^{\frac{3}{2}}}{3} \right]_0^1 \\
&= \frac{2}{3}
\end{aligned}$$

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \sin^2 \left(\frac{k\pi}{12n} \right) &= \int_0^1 \sin^2 \left(\frac{\pi x}{12} \right) dx \\
&= \frac{1}{2} \int_0^1 \left(1 - \cos \left(\frac{\pi x}{6} \right) \right) dx \\
&= \frac{1}{2} \left[x - \frac{6}{\pi} \sin \left(\frac{\pi x}{6} \right) \right]_0^1 \\
&= \frac{1}{2} \left(1 - \frac{6}{\pi} \sin \left(\frac{\pi}{6} \right) \right) \\
&= \frac{1}{2} \left(1 - \frac{3}{\pi} \right)
\end{aligned}$$

$$\begin{aligned}
\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{4n^2 - k^2}} &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{\sqrt{4 - (\frac{k}{n})^2}} \\
&= \int_0^1 \frac{dx}{\sqrt{4 - x^2}} \\
&= \int_0^1 \frac{d(\frac{x}{2})}{\sqrt{1 - (\frac{x}{2})^2}} \\
&= \left[\sin^{-1} \left(\frac{x}{2} \right) \right]_0^1 \\
&= \frac{\pi}{6}
\end{aligned}$$

3. (a)

$$\int_{-1}^{3} \frac{x}{1+x^2}\,dx = \frac{1}{2}\int_{-1}^{3} \frac{d(1+x^2)}{1+x^2} = \frac{1}{2}[\ln(1+x^2)]_{-1}^{3} = \frac{1}{2}(\ln 10 - \ln 2)$$
$$= \frac{\ln 5}{2}$$

(b)

$$\int_{0}^{\frac{\pi}{2}} \sin^3 x \cos^2 x\,dx = -\int_{0}^{\frac{\pi}{2}} \sin^2 x \cos^2 x\, d\cos x$$
$$= -\int_{0}^{\frac{\pi}{2}} (1-\cos^2 x)\cos^2 x\, d\cos x$$
$$= -\int_{0}^{\frac{\pi}{2}} (\cos^2 x - \cos^4 x) d\cos x$$
$$= -\left[\frac{\cos^3 x}{3} - \frac{\cos^5 x}{5}\right]_{0}^{\frac{\pi}{2}}$$
$$= \frac{1}{3} - \frac{1}{5}$$
$$= \frac{2}{15}$$

(c)

$$\int_{0}^{\frac{\pi}{6}} x^2 \cos x\,dx = \int_{0}^{\frac{\pi}{6}} x^2 d\sin x = [x^2\sin x]_{0}^{\frac{\pi}{6}} - \int_{0}^{\frac{\pi}{6}} \sin x dx^2$$
$$= \frac{\pi^2}{72} - 2\int_{0}^{\frac{\pi}{6}} x\sin x dx$$
$$= \frac{\pi^2}{72} + 2\int_{0}^{\frac{\pi}{6}} x d\cos x$$
$$= \frac{\pi^2}{72} + 2[x\cos x]_{0}^{\frac{\pi}{6}} - 2\int_{0}^{\frac{\pi}{6}} \cos x dx$$
$$= \frac{\pi^2}{72} + \frac{\sqrt{3}\pi}{6} - 2[\sin x]_{0}^{\frac{\pi}{6}}$$
$$= \frac{\pi^2}{72} + \frac{\sqrt{3}\pi}{6} - 1$$

(d)

$$\begin{aligned}
\int_0^1 e^{\sqrt{x}} dx &= 2\int_0^1 \sqrt{x} e^{\sqrt{x}} d\sqrt{x} \\
&= 2\int_0^1 \sqrt{x} de^{\sqrt{x}} \\
&= 2[\sqrt{x} e^{\sqrt{x}}]_0^1 - 2\int_0^1 e^{\sqrt{x}} d\sqrt{x} \\
&= 2e - 2[e^{\sqrt{x}}]_0^1 \\
&= 2e - 2[e-1] \\
&= 2
\end{aligned}$$

(e)

$$\begin{aligned}
\int_{-2}^5 |x^2-4| dx &= \int_{-2}^2 (-(x^2-4))dx + \int_2^4 (x^2-4)dx \\
&= \left[4x - \frac{x^3}{3}\right]_{-2}^2 + \left[\frac{x^3}{3} - 4x\right]_2^5 \\
&= \left[\left(8 - \frac{8}{3}\right) - \left((-8 - \frac{-8}{3}\right)\right] + \left[\left(\frac{125}{3} - 20\right) - \left(\frac{8}{3} - 8\right)\right] \\
&= \frac{113}{3}
\end{aligned}$$

(f)

$$\begin{aligned}
\int_{-1}^2 e^{|x-1|} dx &= \int_{-1}^1 e^{1-x} dx + \int_1^2 e^{x-1} dx \\
&= [-e^{1-x}]_{-1}^1 + [e^{x-1}]_1^2 \\
&= [-1 + e^2] + [e-1] \\
&= e^2 + e - 2
\end{aligned}$$

4. (a) $\dfrac{d}{dx}\displaystyle\int_0^x e^{t^2} dt = e^{x^2}$

(b) $\dfrac{d}{dx}\displaystyle\int_1^{x^2} \sqrt{1+e^t}\, dt = 2x\sqrt{1+e^{x^2}}$

(c)

$$\begin{aligned}\frac{d}{dx}\int_x^{\sqrt{x}} \ln(1+t^2)dt &= \ln(1+(\sqrt{x})^2)\left(\frac{1}{2\sqrt{x}}\right) - \ln(1+x^2) \\ &= \frac{\ln(1+x) - 2\sqrt{x}\ln(1+x^2)}{2\sqrt{x}}\end{aligned}$$

(d)

$$\begin{aligned}\frac{d}{dx}\int_{-\sqrt{x}}^{x} \sin(t^2)dt &= \sin(x^2) - \sin(-\sqrt{x})^2\left(-\frac{1}{2\sqrt{x}}\right) \\ &= \sin(x^2) + \frac{\sin x}{2\sqrt{x}}\end{aligned}$$

(e)

$$\begin{aligned}\frac{d}{dx}\int_0^x x\sin(x^2t^2)dt &= \frac{d}{dx}\int_0^x \sin(x^2t^2)d(xt) \\ &= \frac{d}{dx}\int_0^{x^2} \sin u^2 du \\ &= \sin(x^2)^2(2x) \\ &= 2x\sin(x^4)\end{aligned}$$

(f)

$$\begin{aligned}\frac{d}{dx}\int_1^x \frac{e^{xt}dt}{t} &= \frac{d}{dx}\int_1^x \frac{e^{xt}d(xt)}{xt} \\ &= \frac{d}{dx}\int_x^{x^2} \frac{e^u du}{u} \\ &= \frac{e^{x^2}}{x^2}(2x) - \frac{e^x}{x} \\ &= \frac{2e^{x^2} - e^x}{x}\end{aligned}$$

5. (a)

$$\begin{aligned}\lim_{x\to 0}\frac{\int_0^x t\ln(1+t^2)dt}{x^4} &= \lim_{x\to 0}\frac{x\ln(1+x^2)}{4x^3} \\ &= \lim_{x\to 0}\frac{\ln(1+x^2)}{4x^2} \\ &= \frac{1}{4}\end{aligned}$$

(b)

$$\begin{aligned}\lim_{x\to 0}\int_0^x \frac{e^{t^2}-1}{x^3}\,dt &= \lim_{x\to 0}\frac{\int_0^x (e^{t^2}-1)dt}{x^3} \\ &= \lim_{x\to 0}\frac{e^{x^2}-1}{3x^2} \\ &= \frac{1}{3}\end{aligned}$$

(c)

$$\begin{aligned}\lim_{x\to 0}\frac{\int_0^x \sin 6x\sin(t^2)dt}{x^4} &= \lim_{x\to 0}\frac{\sin 6x\int_0^x \sin(t^2)dt}{x^4} \\ &= \lim_{x\to 0}\left(\frac{\sin 6x}{x}\right)\left(\frac{\int_0^x \sin(t^2)dt}{x^3}\right) \\ &= 6\lim_{x\to 0}\frac{\sin(x^2)}{3x^2} \\ &= 2\end{aligned}$$

(d)

$$\begin{aligned}\lim_{x\to 0}\frac{\int_0^x (e^{x^2t^2}-1)dt}{x^5} &= \lim_{x\to 0}\frac{\int_0^x (e^{x^2t^2}-1)d(xt)}{x^6} \\ &= \lim_{x\to 0}\frac{\int_0^{x^2} (e^{u^2}-1)du}{x^6} \\ &= \lim_{x\to 0}\frac{2xe^{x^4}-1}{6x^5} \\ &= \frac{1}{3}\end{aligned}$$