MATH1010 University Mathematics
Limits of functions

1. Evaluate the following limits

(a) $\lim_{x\to 2} \frac{x^5-32}{x^3-8}$

(b) $\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{x+2x^2}\right)$

(c) $\lim_{t\to 0} \left(\frac{t}{\sqrt{x+t}-\sqrt{x}}\right)$

(d) $\lim_{x\to u} \frac{\sqrt{x}-\sqrt{u}}{x-u}$

2. Evaluate the following limits

(a) $\lim_{x\to 0} \frac{\sin 3x}{4x}$

(b) $\lim_{x\to 0} \frac{x^2}{1-\cos x}$

(c) $\lim_{x\to \frac{\pi}{2}} (\frac{\pi}{2}-x)\sec x$

(d) $\lim_{x\to 1} \frac{\sqrt{x}-1}{\ln x}$

(e) $\lim_{x\to 0} \frac{e^{7x}-e^{3x}}{\ln(1+2x)}$

(f) $\lim_{x\to 0} \frac{\sin 9x - \sin 3x}{\sin 2x}$

(g) $\lim_{x\to 0} \frac{\cos x - \cos 3x}{1-\cos 2x}$

(h) $\lim_{x\to 0} \frac{\ln(2-\cos^2 x)}{x^2}$

3. Evaluate the following limits.

(a) $\lim_{x\to +\infty} \frac{2x^3-5x+3}{3x^3-x^2+6}$

(b) $\lim_{x\to +\infty} \frac{\sqrt{9x^4-3x+2}}{x^2-2x+5}$

(c) $\lim_{x\to +\infty} \frac{5e^x+3x^4}{3e^{3x}-2x^3}$

(d) $\lim_{x\to +\infty} \left(\sqrt{4x^2+5x}-2x\right)$

(e) $\lim_{x\to -\infty} \left(x+\sqrt{x^2-4x}\right)$

(f) $\lim_{x\to -\infty} \frac{3e^x+4x^3}{5e^x+x^3}$

(g) $\lim_{x\to +\infty} \frac{\ln(x^2+x-1)}{\ln(x^8-x+1)}$

(h) $\lim_{x\to +\infty} \frac{\ln(8^x+1)}{\ln(2^x+1)}$

(i) $\lim_{x\to 0^+} (1+2x)^{\frac{3}{x}}$

(j) $\lim_{x\to +\infty} \left(\frac{x-3}{x+1}\right)^{2x+5}$

4. Evaluate the following limits.

(a) $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x^3}\right)$

(b) $\lim_{x\to +\infty} \frac{\sin x}{x}$

(c) $\lim_{x\to +\infty} \frac{e^{\cos x}}{1+\ln x}$

(d) $\lim_{x\to +\infty} \frac{\ln(1+ex)-\ln x}{\ln(1+x)}$

5. Evaluate the following limits.

(a) Let $f(x) = \begin{cases} x^2-9, & \text{if } x<4 \\ 5, & \text{if } x=4 \\ \frac{4-x}{2-\sqrt{x}}, & \text{if } x>4 \end{cases}$.

Find $\lim_{x\to 4^-} f(x)$ and $\lim_{x\to 4^+} f(x)$.

(b) Let $f(x) = e^{\frac{1}{x}}$. Find $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^+} f(x)$.

(c) Let $f(x) = \frac{x^2+x-2}{|x-1|}$. Find $\lim_{x\to 1^-} f(x)$ and $\lim_{x\to 1^+} f(x)$.

(d) Let $f(x) = \frac{2x+|x|}{2x-|x|}$. Find $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^+} f(x)$.

Solution:

1. (a)

$$\begin{aligned}\lim_{x\to 2}\frac{x^5-32}{x^3-8} &= \lim_{x\to 2}\frac{(x-2)(x^4+2x^3+4x^2+8x+16)}{(x-2)(x^2+2x+4)} \\ &= \lim_{x\to 2}\frac{x^4+2x^3+4x^2+8x+16}{x^2+2x+4} \\ &= \frac{80}{12} \\ &= \frac{20}{3}\end{aligned}$$

(b)

$$\begin{aligned}\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{x+2x^2}\right) &= \lim_{x\to 0}\frac{1+2x-1}{x+2x^2} \\ &= \lim_{x\to 0}\frac{2}{1+2x} \\ &= 2\end{aligned}$$

(c)

$$\begin{aligned}
\lim_{t\to 0}\left(\frac{t}{\sqrt{x+t}-\sqrt{x}}\right) &= \lim_{t\to 0}\frac{t(\sqrt{x+t}+\sqrt{x})}{(\sqrt{x+t}-\sqrt{x})(\sqrt{x+t}+\sqrt{x})} \\
&= \lim_{t\to 0}\frac{t(\sqrt{x+t}+\sqrt{x})}{x+t-x} \\
&= \lim_{t\to 0}(\sqrt{x+t}+\sqrt{x}) \\
&= 2\sqrt{x}
\end{aligned}$$

(d)

$$\begin{aligned}
\lim_{x\to u}\frac{\sqrt{x}-\sqrt{u}}{x-u} &= \lim_{x\to u}\frac{(\sqrt{x}-\sqrt{u})(\sqrt{x}+\sqrt{u})}{(x-u)(\sqrt{x}+\sqrt{u})} \\
&= \lim_{x\to u}\frac{x-u}{(x-u)(\sqrt{x}+\sqrt{u})} \\
&= \lim_{x\to u}\frac{1}{\sqrt{x}+\sqrt{u}} \\
&= \frac{1}{2\sqrt{u}}
\end{aligned}$$

2. (a)

$$\begin{aligned}
\lim_{x\to 0}\frac{\sin 3x}{4x} &= \lim_{x\to 0}\frac{\sin 3x}{3x}\cdot\frac{3}{4} \\
&= \frac{3}{4}
\end{aligned}$$

(b)

$$\begin{aligned}
\lim_{x\to 0}\frac{x^2}{1-\cos x} &= \lim_{x\to 0}\frac{x^2(1+\cos x)}{(1-\cos x)(1+\cos x)} \\
&= \lim_{x\to 0}\frac{x^2(1+\cos x)}{1-\cos^2 x} \\
&= \lim_{x\to 0}\frac{x^2(1+\cos x)}{\sin^2 x} \\
&= \lim_{x\to 0}\left(\frac{x}{\sin x}\right)^2(1+\cos x) \\
&= 2
\end{aligned}$$

(c)

$$\begin{aligned}\lim_{x\to\frac{\pi}{2}}(x-\frac{\pi}{2})\sec x &= \lim_{x\to\frac{\pi}{2}}\frac{\frac{\pi}{2}-x}{\cos x}\\ &= \lim_{y\to 0}\frac{y}{\cos(\frac{\pi}{2}-y)} \quad (y=\frac{\pi}{2}-x)\\ &= \lim_{y\to 0}\frac{y}{\sin y}\\ &= 1\end{aligned}$$

(d)

$$\begin{aligned}\lim_{x\to 1}\frac{\sqrt{x}-1}{\ln x} &= \lim_{y\to 0}\frac{\sqrt{1+y}-1}{\ln(1+y)} \quad (x=1+y)\\ &= \lim_{y\to 0}\frac{(\sqrt{1+y}-1)(\sqrt{1+y}+1)}{(\sqrt{1+y}+1)\ln(1+y)}\\ &= \lim_{y\to 0}\frac{1+y-1}{(\sqrt{1+y}+1)\ln(1+y)}\\ &= \lim_{y\to 0}\frac{1}{\sqrt{1+y}+1}\cdot\frac{y}{\ln(1+y)}\\ &= \frac{1}{2}\end{aligned}$$

(e)

$$\begin{aligned}\lim_{x\to 0}\frac{e^{7x}-e^{3x}}{\ln(1+2x)} &= \lim_{x\to 0}2e^{3x}\cdot\frac{e^{4x}-1}{4x}\cdot\frac{2x}{\ln(1+2x)}\\ &= 2\end{aligned}$$

(f)

$$\begin{aligned}\lim_{x\to 0}\frac{\sin 9x-\sin 3x}{\sin 2x} &= \lim_{x\to 0}\frac{\frac{9\sin 9x}{9x}-\frac{3\sin 3x}{3x}}{\frac{2\sin 2x}{2x}}\\ &= \frac{9-3}{2}\\ &= 3\end{aligned}$$

(g)

$$\begin{aligned}\lim_{x\to 0}\frac{\cos x-\cos 3x}{1-\cos 2x} &= \lim_{x\to 0}\frac{2\sin 2x\sin x}{2\sin^2 x}\\ &= \lim_{x\to 0}\frac{2\sin x\cos x}{\sin x}\\ &= \lim_{x\to 0} 2\cos x\\ &= 2\end{aligned}$$

(h)

$$\begin{aligned}\lim_{x\to 0}\frac{\ln(2-\cos^2 x)}{x^2} &= \lim_{x\to 0}\frac{\ln(1+\sin^2 x)}{x^2}\\ &= \lim_{x\to 0}\frac{\ln(1+\sin^2 x)}{\sin^2 x}\cdot\frac{\sin^2 x}{x^2}\\ &= \lim_{y\to 0}\frac{\ln(1+y)}{y}\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2\\ &= 1\end{aligned}$$

3. (a)

$$\begin{aligned}\lim_{x\to+\infty}\frac{2x^3-5x+3}{3x^3-x^2+6} &= \lim_{x\to+\infty}\frac{2-\frac{5}{x^2}+\frac{3}{x^3}}{3-\frac{1}{x}+\frac{6}{x^3}}\\ &= \frac{2}{3}\end{aligned}$$

(b)

$$\begin{aligned}\lim_{x\to+\infty}\frac{\sqrt{9x^4-3x+2}}{x^2-2x+5} &= \lim_{x\to+\infty}\frac{\sqrt{9-\frac{3}{x^3}+\frac{2}{x^4}}}{1-\frac{2}{x}+\frac{5}{x^2}}\\ &= 3\end{aligned}$$

(c)

$$\begin{aligned}\lim_{x\to+\infty}\frac{5e^x+3x^4}{3e^{3x}-2x^3} &= \lim_{x\to+\infty}\frac{5e^{-2x}+3x^4e^{-3x}}{3-2x^3e^{-3x}}\\ &= 0\end{aligned}$$

(d)

$$\begin{aligned}
\lim_{x\to+\infty}\left(\sqrt{4x^2+5x}-2x\right) &= \lim_{x\to+\infty}\frac{(\sqrt{4x^2+5x}-2x)(\sqrt{4x^2+5x}+2x)}{(\sqrt{4x^2+5x}+2x)} \\
&= \lim_{x\to+\infty}\frac{4x^2+5x-4x^2}{\sqrt{4x^2+5x}+2x} \\
&= \lim_{x\to+\infty}\frac{5}{(\sqrt{4+\frac{5}{x}}+2)} \\
&= \frac{5}{4}
\end{aligned}$$

(e)

$$\begin{aligned}
\lim_{x\to-\infty}\left(x+\sqrt{x^2-4x}\right) &= \lim_{x\to-\infty}\frac{(x+\sqrt{x^2-4x})(x-\sqrt{x^2-4x})}{x-\sqrt{x^2-4x}} \\
&= \lim_{x\to-\infty}\frac{x^2-(x^2-4x)}{x-\sqrt{x^2-4x}} \\
&= \lim_{x\to-\infty}\frac{4x}{x-\sqrt{x^2-4x}} \\
&= \lim_{x\to-\infty}\frac{4}{1+\sqrt{1-\frac{4}{x}}} \quad \text{(Note that } x=-\sqrt{x^2} \text{ when } x<0.\text{)} \\
&= 2
\end{aligned}$$

(f)

$$\begin{aligned}
\lim_{x\to-\infty}\frac{3e^x+4x^3}{5e^x+x^3} &= \lim_{x\to-\infty}\frac{3x^{-3}e^x+4}{5x^{-3}e^x+1} \\
&= 4
\end{aligned}$$

(g)

$$\begin{aligned}\lim_{x\to+\infty}\frac{\ln(x^2+x-1)}{\ln(x^8-x+1)} &= \lim_{x\to+\infty}\frac{\ln(x^2(1+\frac{1}{x}-\frac{1}{x^2}))}{\ln(x^8(1-\frac{1}{x^7}+\frac{1}{x^8}))}\\ &= \lim_{x\to+\infty}\frac{2\ln x+\ln(1+\frac{1}{x}-\frac{1}{x^2})}{8\ln x+\ln(1-\frac{1}{x^7}+\frac{1}{x^8})}\\ &= \lim_{x\to+\infty}\frac{2+\frac{\ln(1+\frac{1}{x}-\frac{1}{x^2})}{\ln x}}{8+\frac{\ln(1-\frac{1}{x^7}+\frac{1}{x^8})}{\ln x}}\\ &= \frac{1}{4}\end{aligned}$$

(h)

$$\begin{aligned}\lim_{x\to+\infty}\frac{\ln(8^x+1)}{\ln(2^x+1)} &= \lim_{x\to+\infty}\frac{\ln(8^x(1+8^{-x}))}{\ln(2^x(1+2^{-x}))}\\ &= \lim_{x\to+\infty}\frac{x\ln 8+\ln(1+8^{-x})}{x\ln 2+\ln(1+2^{-x})}\\ &= \lim_{x\to+\infty}\frac{\ln 8+\frac{\ln(1+8^{-x})}{x}}{\ln 2+\frac{\ln(1+2^{-x})}{x}}\\ &= \frac{\ln 8}{\ln 2}\\ &= \frac{3\ln 2}{\ln 2}\\ &= 3\end{aligned}$$

(i)

$$\lim_{x\to0^+}(1+2x)^{\frac{3}{x}} = \lim_{y\to+\infty}\left(1+\frac{2}{y}\right)^{3y} = (e^2)^3 = e^6 \quad (y=\frac{1}{x})$$

(j)

$$\begin{aligned}\lim_{x\to+\infty}\left(\frac{x-3}{x+1}\right)^{2x+5} &= \lim_{x\to+\infty}\left(1-\frac{4}{x+1}\right)^{2(x+1)+3}\\ &= \lim_{y\to+\infty}\left(1-\frac{4}{y}\right)^{2y}\lim_{x\to+\infty}\left(1-\frac{4}{x+1}\right)^{3} \quad (y=x+1)\\ &= (e^{-4})^2\\ &= e^{-8}\end{aligned}$$

4. (a) Since $|\sin\left(\frac{1}{x^3}\right)| \leq 1$ is bounded and $\lim_{x\to 0} x^2 = 0$, we have

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x^3}\right) = 0$$

(b) Since $|\sin x| \leq 1$ is bounded and $\lim_{x\to+\infty} \frac{1}{x} = 0$, we have

$$\lim_{x\to+\infty} \frac{\sin x}{x} = 0$$

(c) Since $|e^{\cos x}| \leq e$ is bounded and $\lim_{x\to+\infty} \frac{1}{1+\ln x} = 0$, we have

$$\lim_{x\to+\infty} \frac{e^{\cos x}}{1+\ln x} = 0$$

(d) Since $|\ln(1+ex) - \ln x| = |\ln(\frac{1}{x}+e)| \leq \ln(1+e)$ is bounded when $x > 1$ and $\lim_{x\to+\infty} \frac{1}{\ln(1+x)} = 0$, we have

$$\lim_{x\to+\infty} \frac{\ln(1+ex) - \ln x}{\ln(1+x)} = 0$$

5. (a) $\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} (x^2 - 9) = 7$

$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} \frac{4-x}{2-\sqrt{x}} = \lim_{x\to 4^+} (2+\sqrt{x}) = 4$

(b) $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} e^{\frac{1}{x}} = \lim_{y\to+\infty} e^{-y} = 0$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{\frac{1}{x}} = \lim_{y\to+\infty} e^{y}$ does not exist.

(c) $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{(x+2)(x-1)}{-(x-1)} = \lim_{x\to 2^-} (-(x+2)) = -3$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{(x+2)(x-1)}{x-1} = \lim_{x\to 2^+} (x+2) = 3$

(d) $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{2x-x}{2x-(-x)} = \lim_{x\to 0^-} \frac{x}{3x} = \frac{1}{3}$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{2x+x}{2x-x} = \lim_{x\to 0^+} \frac{3x}{x} = 3$