MMAT5520 Differential Equations & Linear Algebra
Mid-term Exam (27 Oct 2016)
Suggested Solution
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Name:____________________________ID:____________________ Marks:__________/50

Answer all questions.

Write your answers in the space provided.

1. (8 marks) Let $A = \begin{pmatrix} a & 3 & 1 \\ 1 & 3 & 0 \\ -2 & 2 & -3 \end{pmatrix}$. It is known that $\det(A) = -1$.

   (a) Find a.

   (b) Find the determinant of $2A^T$.

   (c) Find $\mathbf{A}^{-1}$.

   Solution:

   (a) Expanding the determinant along the second row, we have

$$\begin{aligned} \det(\mathbf{A}) &= -1 \\ -\left((1)\begin{vmatrix} 3 & 1 \\ 2 & -3 \end{vmatrix} - (3)\begin{vmatrix} a & 1 \\ -2 & -3 \end{vmatrix}\right) &= -1 \\ -\left((1)(-11) - (3)(-3a+2)\right) &= -1 \\ -(-11 + 9a - 6) &= -1 \\ 17 - 9a &= -1 \\ -9a &= -18 \\ a &= 2. \end{aligned}$$

   (b) Since $2\mathbf{A}^T$ is obtained from multiplying the first, second, and the third row of $\mathbf{A}^T$ by a constant 2, we have

$$\begin{aligned} \det(2\mathbf{A}^T) &= 2^3 \det(\mathbf{A}^T) \\ &= 8 \det(\mathbf{A}^T) \\ &= 8 \det(\mathbf{A}) \\ &= 8(-1) \\ &= -8. \end{aligned}$$

(c) The adjoint matrix of $\mathbf{A}$ is given by

$$\text{adj}\mathbf{A} = \begin{pmatrix} \begin{vmatrix} 3 & 0 \\ 2 & -3 \end{vmatrix} & -\begin{vmatrix} 3 & 1 \\ 2 & -3 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 3 & 0 \end{vmatrix} \\ -\begin{vmatrix} 1 & 0 \\ -2 & -3 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ -2 & -3 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} \\ \begin{vmatrix} 1 & 3 \\ -2 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ -2 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} -9 & 11 & -3 \\ 3 & -4 & 1 \\ 8 & -10 & 3 \end{pmatrix}.$$

Therefore

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\text{adj}\mathbf{A} = \frac{1}{-1}\begin{pmatrix} -9 & 11 & -3 \\ 3 & -4 & 1 \\ 8 & -10 & 3 \end{pmatrix} = \begin{pmatrix} 9 & -11 & 3 \\ -3 & 4 & -1 \\ -8 & 10 & -3 \end{pmatrix}.$$

□

2. (6 marks) Solve

$$\frac{dy}{dx} = \frac{e^x}{2y}, \quad y(0) = 2$$

Solution: The differential equation is a separable equation.

$$\begin{aligned} \frac{dy}{dx} &= \frac{e^x}{2y} \\ 2y\,dy &= e^x\,dx \\ \int 2y\,dy &= \int e^x\,dx \\ y^2 &= e^x + C. \end{aligned}$$

By the initial condition $y(0) = 2 > 0$, we choose the positive branch

$$y = (e^x + C)^{\frac{1}{2}}.$$

Moreover, we have

$$4 = 2^2 = e^0 = 1 + C \iff C = 3.$$

Therefore the solution is

$$y = (e^x + 3)^{\frac{1}{2}}.$$

□

3. (8 marks) Solve the equation

$$\frac{dy}{dx} - y = xy^5.$$

Solution: The differential equation is a Bernoulli's equation. Let $u = y^{1-5} = y^{-4}$. We have

$$\begin{aligned}
\frac{du}{dx} &= (1-5)\,y^{-5}\frac{dy}{dx} \\
\frac{du}{dx} &= -4y^{-5}\left(y + xy^5\right) \\
\frac{du}{dx} &= -4y^{-4} - 4x \\
\frac{du}{dx} + 4u &= -4x.
\end{aligned}$$

An integrating factor is

$$\exp\left(\int 4\,dx\right) = e^{4x}.$$

Multiplying e^{4x} on both sides, we ahve

$$\begin{aligned}
e^{4x}\frac{du}{dx} + 4e^{4x}u &= -4xe^{4x} \\
\frac{d}{dx}(e^{4x}u) &= -4xe^{4x} \\
e^{4x}u &= \int (-4xe^{4x})\,dx.
\end{aligned}$$

Using integration by parts, we have

$$\begin{aligned}
\int (-4xe^{4x})\,dx &= -\int x\,d(e^{4x}) \\
&= -xe^{4x} + \int e^{4x}\,dx \\
&= -xe^{4x} + \frac{1}{4}e^{4x} + C.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
e^{4x}u &= -xe^{4x} + \frac{1}{4}e^{4x} \\
u &= -x + \frac{1}{4} + Ce^{-4x} \\
y^{-4} &= -x + \frac{1}{4} + Ce^{-4x} \\
y^4 &= \left(-x + \frac{1}{4} + Ce^{-4x}\right)^{-1} \text{ or } y = 0.
\end{aligned}$$

□

4. (8 marks) Show that the equation

$$(4xy + 2y^2)dx + (x^2 + 3xy)dy = 0$$

has an integrating factor of the form $\mu(x, y) = y^k$ and solve the equation.
Solution: Multiplying the equation by $\mu(x, y) = y^k$, we have

$$(4xy^{k+1} + 2y^{k+2})dx + (x^2y^k + 3xy^{k+1})dy = 0$$

Now we have

$$\begin{cases} M(x,y) = 4xy^{k+1} + 2y^{k+2} \\ N(x,y) = x^2y^k + 3xy^{k+1} \end{cases} \implies \begin{cases} \dfrac{\partial M}{\partial y} = 4(k+1)xy^k + 2(k+2)y^{k+1} \\ \\ \dfrac{\partial N}{\partial x} = 2xy^k + 3y^{k+1} \end{cases}.$$

By choosing $k = -\dfrac{1}{2}$, we have

$$\frac{\partial M}{\partial y} = 2xy^{-\frac{1}{2}} + 3y^{\frac{1}{2}} = \frac{\partial N}{\partial x},$$

and the equation is exact. In this case, we set

$$\begin{aligned} F(x,y) &= \int M(x,y)\,dx \\ &= \int (4xy^{\frac{1}{2}} + 2y^{\frac{3}{2}})\,dx \\ &= 2x^2y^{\frac{1}{2}} + 2xy^{\frac{3}{2}} + g(y). \end{aligned}$$

Now, we want

$$\begin{aligned} \frac{\partial F}{\partial y} &= N(x,y) \\ x^2y^{-\frac{1}{2}} + 3xy^{\frac{1}{2}} + g'(y) &= x^2y^{-\frac{1}{2}} + 3xy^{\frac{1}{2}} \\ g'(y) &= 0 \\ g(y) &= C. \end{aligned}$$

The general solution of the differential equation is

$$F(x,y) = C \iff x^2y^{\frac{1}{2}} + xy^{\frac{3}{2}} = C.$$

□

5. (6 marks) Let C be the circle determined by the points $(3,5)$, $(2,-2)$ and $(4,2)$. Write down the equation of C in the form $x^2+y^2+Dx+Ey+F=0$ by considering a suitable determinant. (Finding D,E,F by solving equations would obtain zero mark.)

   Solution: The required equation is

$$\begin{vmatrix} 1 & x & y & x^2+y^2 \\ 1 & 3 & 5 & 3^2+5^2 \\ 1 & 2 & -2 & 2^2+(-2)^2 \\ 1 & 4 & 2 & 4^2+2^2 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & x & y & x^2+y^2 \\ 1 & 3 & 5 & 34 \\ 1 & 2 & -2 & 8 \\ 1 & 4 & 2 & 20 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & x & y & x^2+y^2 \\ 1 & 3 & 5 & 34 \\ 0 & -1 & -7 & -26 \\ 0 & 1 & -3 & -14 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & x & y & x^2+y^2 \\ 1 & 3 & 5 & 34 \\ 0 & 1 & 7 & 26 \\ 0 & 1 & -3 & -14 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & x & y & x^2+y^2 \\ 1 & 0 & -16 & -44 \\ 0 & 1 & 7 & 26 \\ 0 & 0 & -10 & -40 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & x & y & x^2+y^2 \\ 1 & 0 & -16 & -44 \\ 0 & 1 & 7 & 26 \\ 0 & 0 & 1 & 4 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & x & y & x^2+y^2 \\ 1 & 0 & 0 & 20 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{vmatrix} = 0.$$

   Expanding the determinant along the second row, we have

$$-\left((1)\begin{vmatrix} x & y & x^2+y^2 \\ 1 & 0 & -2 \\ 0 & 1 & 4 \end{vmatrix} - (20)\begin{vmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}\right) = 0$$

$$\begin{aligned} -\left(x\begin{vmatrix} 0 & -2 \\ 1 & 4 \end{vmatrix} - y\begin{vmatrix} 1 & -2 \\ 0 & 4 \end{vmatrix} + (x^2+y^2)\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\right) + (20)(1) &= 0 \\ -(2x-4y+(x^2+y^2))+20 &= 0 \\ -x^2-y^2-2x+4y+20 &= 0 \\ x^2+y^2+2x-4y-20 &= 0. \end{aligned}$$

□

6. (8 marks) Let $A = \begin{pmatrix} 1 & -3 & 2 & 3 & 4 \\ -2 & 6 & -4 & -1 & -3 \\ 3 & -9 & 6 & 2 & 5 \end{pmatrix}$.

   (a) Find a basis for the null space of A.

   (b) Find a basis for the row space of A.

   (c) Find a basis for the column space of A.

   (d) Suppose B is a matrix which is row equivalent to A. Find the rank and nullity of B^T.

Solution:

$$\mathbf{A} = \begin{pmatrix} 1 & -3 & 2 & 3 & 4 \\ -2 & 6 & -4 & -1 & -3 \\ 3 & -9 & 6 & 2 & 5 \end{pmatrix}$$

$$\xrightarrow{\substack{R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 - 3R_1}} \begin{pmatrix} 1 & -3 & 2 & 3 & 4 \\ 0 & 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & -7 & -7 \end{pmatrix}$$

$$\xrightarrow{R_2 \to \frac{1}{5}R_2} \begin{pmatrix} 1 & -3 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -7 & -7 \end{pmatrix}$$

$$\xrightarrow{\substack{R_1 \to R_1 - 3R_2 \\ R_3 \to R_3 + 7R_2}} \begin{pmatrix} 1 & -3 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We arrive at the reduced row echelon form of $\mathbf{A}$. The first and fourth columns contain leading entries. The second, third and fifth columns correspond to free variables.

(a) The set

$$\{(3,1,0,0,0)^T, (-2,0,1,0,0)^T, (-1,0,0,-1,1)^T\}$$

constitutes a basis for Null($\mathbf{A}$).

(b) The set

$$\{(1,-3,2,0,1), (0,0,0,1,1)\}$$

constitutes a basis for Row($\mathbf{A}$).

(c) The set

$$\{(1,-2,3)^T, (3,-1,2)^T\}$$

constitutes a basis for Col($\mathbf{A}$).

(d) Since $\mathbf{B}$ is row equivalent to $\mathbf{A}$, we have $\mathbf{Bx} = \mathbf{0}$ if and only if $\mathbf{Ax} = \mathbf{0}$. Hence

$$\text{Null}(\mathbf{B}) = \text{Null}(\mathbf{A}) \implies \text{nullity}(\mathbf{B}) = \text{nullity}(\mathbf{A}) = 3.$$

Note that $\mathbf{B}$ is a 3×5 matrix. By rank-nullity theorem, we have

$$\text{rank}(\mathbf{B}) + \text{nullity}(\mathbf{B}) = 5 \implies \text{rank}(\mathbf{B}) = 2.$$

This further implies

$$\text{rank}(\mathbf{B}^T) = \text{rank}(\mathbf{B}) = 2.$$

Note that $\mathbf{B}^T$ is a 5×3 matrix. Again, by rank-nullity theorem, we have

$$\text{rank}(\mathbf{B}^T) + \text{nullity}(\mathbf{B}^T) = 3 \implies \text{nullity}(\mathbf{B}^T) = 1.$$

□

7. (6 marks) State whether the following statements are true or false. No explanation is required.

   (a) Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^3$. If $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \neq \mathbb{R}^3$, then $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent.

   (b) If $V = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, then $\dim(V) = 3$.

   (c) If $\mathbf{v}_3 \notin \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$, then $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent.

   (d) If $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent, then $\mathbf{u}_1 = \mathbf{v}_1, \mathbf{u}_2 = \mathbf{v}_1 + \mathbf{v}_2$, $\mathbf{u}_3 = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$ are linearly independent.

   (e) For any vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, the vectors $\mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_2 - \mathbf{v}_3, \mathbf{v}_3 - \mathbf{v}_1$ are linearly dependent.

   (f) If $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent, then $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} \neq \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$

Solution:

Question	(a)	(b)	(c)	(d)	(e)	(f)
Answer	True	False	False	True	True	False

— End of Paper —