MATH1010 University Mathematics

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1. Sequences
   - Limits of sequences
   - Squeeze theorem
   - Monotone convergence theorem

2. Limits and Continuity
   - Exponential, logarithmic and trigonometric functions
   - Limits of functions
   - Continuity of functions

Definition (Infinite sequence of real numbers)

An **infinite sequence of real numbers** is defined by a function from the set of positive integers $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ to the set of real numbers $\mathbb{R}$.

Example (Sequences)

- Arithmetic sequence: $a_n = 3n + 4$; $7, 10, 13, 16 \ldots$
- Geometric sequence: $a_n = 3 \cdot 2^n$; $6, 12, 24, 48 \ldots$
- Fibonacci's sequence:
  $$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right);$$
  $1, 1, 2, 3, 5, 8, 13, \ldots$

Definition (Limit of sequence)

1. Suppose there exists real number L such that for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $n > N$, we have $|a_n - L| < \epsilon$. Then we say that a_n is **convergent**, or a_n **converges to** L, and write

$$\lim_{n\to\infty} a_n = L.$$

   Otherwise we say that a_n is **divergent**.

2. Suppose for any $M > 0$, there exists $N \in \mathbb{N}$ such that for any $n > N$, we have $a_n > M$. Then we say that a_n **tends to** $+\infty$ as n tends to infinity, and write

$$\lim_{n\to\infty} a_n = +\infty.$$

   We define a_n **tends to** $-\infty$ in a similar way. Note that a_n is divergent if it tends to $\pm\infty$.

Example (Intuitive meaning of limits of infinite sequences)

a_n	First few terms	Limit
$\frac{1}{n^2}$	$1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots$	0
$\frac{n}{n+1}$	$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$	1
$(-1)^{n+1}$	$1, -1, 1, -1, \ldots$	does not exist
$2n$	$2, 4, 6, 8, \ldots$	does not exist$/+\infty$
$\left(1+\frac{1}{n}\right)^n$	$2, \frac{9}{4}, \frac{64}{27}, \frac{625}{256}, \ldots$	$e \approx 2.71828$
$\frac{F_{n+1}}{F_n}$	$1, 2, \frac{3}{2}, \frac{5}{3}, \ldots$	$\frac{1+\sqrt{5}}{2} \approx 1.61803$

Definition (Monotonic sequence)

1. We say that a_n is **monotonic increasing** (**decreasing**) if for any $m < n$, we have $a_m \leq a_n$ ($a_m \geq a_n$).
2. We say that a_n is **strictly increasing** (**decreasing**) if for any $m < n$, we have $a_m < a_n$ ($a_m > a_n$).

Definition (Bounded sequence)

We say that a_n is **bounded** if there exists real number M such that $|a_n| < M$ for any $n \in \mathbb{N}$.

Example (Bounded and monotonic sequence)

a_n	Bounded	Monotonic	Convergent
$\frac{1}{n^2}$	✓	✓	✓
$\frac{2n-(-1)^n}{n}$	✓	×	✓
n^2	×	✓	×
$1-(-1)^n$	✓	×	×
$(-1)^n n$	×	×	×

Theorem

If a_n is **convergent**, *then a_n is* **bounded**.

$$\textbf{Convergent} \Rightarrow \textbf{Bounded}$$

Note that the converse of the above statement is not correct.

$$\textbf{Bounded} \not\Rightarrow \textbf{Convergent}$$

The following theorem is very important and we will discuss it in details later.

Theorem (Monotone convergence theorem)

If a_n is **bounded** *and* **monotonic**, *then a_n is* **convergent**.

$$\textbf{Bounded} \textit{ and } \textbf{Monotonic} \Rightarrow \textbf{Convergent}$$

Exercise (True or False)

Suppose $\lim_{n\to\infty} a_n = a$ *and* $\lim_{n\to\infty} b_n = b$. *Then*

$$\lim_{n\to\infty}(a_n \pm b_n) = a \pm b.$$

Answer: **T**

Exercise (True or False)

Suppose $\lim_{n\to\infty} a_n = a$ *and* c *is a real number. Then*

$$\lim_{n\to\infty} ca_n = ca.$$

Answer: **T**

Exercise (True or False)

If $\lim_{n\to\infty} a_n = a$ *and* $\lim_{n\to\infty} b_n = b$, *then*

$$\lim_{n\to\infty} a_n b_n = ab.$$

Answer: **T**

Exercise (True or False)

If $\lim_{n\to\infty} a_n = a$ *and* $\lim_{n\to\infty} b_n = b$, *then*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}.$$

Answer: **F**

Exercise (True or False)

If $\lim_{n\to\infty} a_n = a$ *and* $\lim_{n\to\infty} b_n = b \neq 0$, *then*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}.$$

Answer: **T**

Exercise (True or False)

If $\lim_{n\to\infty} a_n = 0$, *then*

$$\lim_{n\to\infty} a_n b_n = 0.$$

Answer: **F**

Example

For $a_n = \frac{1}{n}$ and $b_n = n$, we have $\lim_{n\to\infty} a_n = 0$ but

$$\lim_{n\to\infty} a_n b_n \neq 0.$$

Exercise (True or False)

If $\lim_{n\to\infty} a_n = 0$ *and* b_n *is* **convergent**, *then*

$$\lim_{n\to\infty} a_n b_n = 0.$$

Answer: **T**

Proof.

$$\begin{aligned}\lim_{n\to\infty} a_n b_n &= \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n \\ &= 0\end{aligned}$$

□

Exercise (True or False)

If $\lim_{n\to\infty} a_n = 0$ *and* b_n *is* **bounded**, *then*

$$\lim_{n\to\infty} a_n b_n = 0.$$

Answer: **T**

Caution! The previous proof does not work.

Exercise (True or False)

If a_n^2 is convergent, then a_n is convergent.

Answer: **F**

Example

For $a_n = (-1)^n$, a_n^2 converges to 1 but a_n is divergent.

Exercise (True or False)

If a_n is convergent, then $|a_n|$ is convergent.

Answer: **T**

Exercise (True or False)

If $|a_n|$ is convergent, then a_n is convergent.

Answer: **F**

Exercise (True or False)

If a_n and b_n are divergent, then $a_n + b_n$ is divergent.

Answer: **F**

Example

The sequences $a_n = n$ and $b_n = -n$ are divergent but $a_n + b_n = 0$ converges to 0.

Exercise (True or False)

If a_n *is convergent and* $\lim_{n\to\infty} b_n = \pm\infty$, *then*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0.$$

Answer: **T**

Exercise (True or False)

If a_n *is* **bounded** *and* $\lim_{n\to\infty} b_n = \pm\infty$, *then*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0.$$

Answer: **T**

Exercise (True or False)

Suppose a_n is bounded. Suppose b_n is a sequence and there exists N such that $b_n = a_n$ for any $n > N$. Then b_n is bounded.

Answer: **T**

Exercise (True or False)

Suppose $\lim_{n\to\infty} a_n = a$. *Suppose* b_n *is a sequence and there exists* N *such that* $b_n = a_n$ *for any* $n > N$. *Then*

$$\lim_{n\to\infty} b_n = a.$$

Answer: **T**

Exercise (True or False)

Suppose a_n and b_n are convergent sequences such that $a_n < b_n$ for any n. Then

$$\lim_{n\to\infty} a_n < \lim_{n\to\infty} b_n.$$

Answer: **F**

Example

There sequences $a_n = 0$ and $b_n = \frac{1}{n}$ satisfy $a_n < b_n$ for any n. However

$$\lim_{n\to\infty} a_n \not< \lim_{n\to\infty} b_n$$

because both of them are 0.

Exercise (True or False)

Suppose a_n *and* b_n *are convergent sequences such that* $a_n \leq b_n$ *for any* n*. Then*

$$\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n.$$

Answer: **T**

Exercise (True or False)

If $\lim_{n\to\infty} a_n = a$, *then*

$$\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} a_{2n+1} = a.$$

Answer: **T**

Exercise (True or False)

If $\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} a_{2n+1} = a$, *then*

$$\lim_{n\to\infty} a_n = a.$$

Answer: **T**

Exercise (True or False)

If a_n is convergent, then

$$\lim_{n\to\infty} (a_{n+1} - a_n) = 0.$$

Answer: **T**

Exercise (True or False)

If $\lim_{n\to\infty}(a_{n+1} - a_n) = 0$, *then* a_n *is convergent.*

Answer: **F**

Example

Let $a_n = \sqrt{n}$. Then $\lim_{n\to\infty}(a_{n+1} - a_n) = 0$ and a_n is divergent.

Exercise (True or False)

If $\lim_{n\to\infty}(a_{n+1} - a_n) = 0$ *and* a_n *is bounded, then* a_n *is convergent.*

Answer: **F**

Example

$$0, \frac{1}{2}, 1, \frac{2}{3}, \frac{1}{3}, 0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1, \frac{4}{5}, \frac{3}{5}, \frac{2}{5}, \frac{1}{5}, 0, \frac{1}{6}, \frac{2}{6}, \ldots$$

Example

Let $a > 0$ be a positive real number.

$$\lim_{n\to\infty} a^n = \begin{cases} +\infty, & \text{if } a > 1 \\ 1, & \text{if } a = 1 \\ 0, & \text{if } 0 < a < 1 \end{cases}.$$

Example

$$\begin{aligned}\lim_{n\to\infty}\frac{2n-5}{3n+1} &= \lim_{n\to\infty}\frac{2-\frac{5}{n}}{3+\frac{1}{n}}\\ &= \frac{2-0}{3+0}\\ &= \frac{2}{3}\end{aligned}$$

Example

$$\begin{aligned}\lim_{n\to\infty}\frac{n^3-2n+7}{4n^3+5n^2-3} &= \lim_{n\to\infty}\frac{1-\frac{2}{n^2}+\frac{7}{n^3}}{4+\frac{5}{n}-\frac{3}{n^3}} \\ &= \frac{1}{4}\end{aligned}$$

Example

$$\begin{aligned}
\lim_{n\to\infty}\frac{3n-\sqrt{4n^2+1}}{3n+\sqrt{9n^2+1}} &= \lim_{n\to\infty}\frac{3-\frac{\sqrt{4n^2+1}}{n}}{3+\frac{\sqrt{9n^2+1}}{n}} \\
&= \lim_{n\to\infty}\frac{3-\sqrt{4+\frac{1}{n^2}}}{3+\sqrt{9+\frac{1}{n^2}}} \\
&= \frac{1}{6}
\end{aligned}$$

Example

$$\begin{aligned}
& \lim_{n\to\infty}(n-\sqrt{n^2-4n+1}) \\
= & \lim_{n\to\infty}\frac{(n-\sqrt{n^2-4n+1})(n+\sqrt{n^2-4n+1})}{n+\sqrt{n^2-4n+1}} \\
= & \lim_{n\to\infty}\frac{n^2-(n^2-4n+1)}{n+\sqrt{n^2-4n+1}} \\
= & \lim_{n\to\infty}\frac{4n-1}{n+\sqrt{n^2-4n+1}} \\
= & \lim_{n\to\infty}\frac{4-\frac{1}{n}}{1+\sqrt{1-\frac{4}{n}+\frac{1}{n^2}}} \\
= & \ 2
\end{aligned}$$

Example

$$\begin{aligned}
\lim_{n\to\infty}\frac{\ln(n^4+1)}{\ln(n^3+1)} &= \lim_{n\to\infty}\frac{\ln(n^4(1+\frac{1}{n^4}))}{\ln(n^3(1+\frac{1}{n^3}))}\\
&= \lim_{n\to\infty}\frac{\ln n^4+\ln(1+\frac{1}{n^4})}{\ln n^3+\ln(1+\frac{1}{n^3})}\\
&= \lim_{n\to\infty}\frac{4\ln n+\ln(1+\frac{1}{n^4})}{3\ln n+\ln(1+\frac{1}{n^3})}\\
&= \lim_{n\to\infty}\frac{4+\frac{\ln(1+\frac{1}{n^4})}{\ln n}}{3+\frac{\ln(1+\frac{1}{n^3})}{\ln n}}\\
&= \frac{4}{3}
\end{aligned}$$

Theorem (Squeeze theorem)

Suppose a_n, b_n, c_n *are sequences such that* $a_n \leq b_n \leq c_n$ *for any* n *and* $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$. *Then* b_n *is convergent and*

$$\lim_{n\to\infty} b_n = L.$$

Theorem

If a_n is bounded and $\lim_{n\to\infty} b_n = 0$, then $\lim_{n\to\infty} a_n b_n = 0$.

Proof.

Since a_n is bounded, there exists M such that $-M < a_n < M$ for any n. Thus

$$-M|b_n| < a_n b_n < M|b_n|$$

for any n. Now

$$\lim_{n\to\infty}(-M|b_n|) = \lim_{n\to\infty} M|b_n| = 0.$$

Therefore by squeeze theorem, we have

$$\lim_{n\to\infty} a_n b_n = 0.$$

□

Example

Find $\lim_{n\to\infty} \frac{\sqrt{n}+(-1)^n}{\sqrt{n}-(-1)^n}$.

Solution

Since $(-1)^n$ *is bounded and* $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, *we have* $\lim_{n\to\infty} \frac{(-1)^n}{\sqrt{n}} = 0$ *and therefore*

$$\begin{aligned} \lim_{n\to\infty} \frac{\sqrt{n}+(-1)^n}{\sqrt{n}-(-1)^n} &= \lim_{n\to\infty} \frac{1+\frac{(-1)^n}{\sqrt{n}}}{1-\frac{(-1)^n}{\sqrt{n}}} \\ &= 1 \end{aligned}$$

Example

Show that $\lim_{n\to\infty} \frac{2^n}{n!} = 0$.

Proof.

Observe that for any $n \geq 3$,

$$0 < \frac{2^n}{n!} = 2\left(\frac{2}{2} \cdot \frac{2}{3} \cdot \frac{2}{4} \cdots \frac{2}{n-1}\right)\frac{2}{n} \leq 2 \cdot \frac{2}{n} = \frac{4}{n}$$

and $\lim_{n\to\infty} \frac{4}{n} = 0$. By squeeze theorem, we have

$$\lim_{n\to\infty} \frac{2^n}{n!} = 0.$$

□

Theorem (Monotone convergence theorem)

If a_n is **bounded** *and* **monotonic**, *then a_n is* **convergent**.

Bounded *and* **Monotonic** $\Rightarrow$ **Convergent**

Example

Let a_n be the sequence defined by the recursive relation

$$\begin{cases} a_{n+1} = \sqrt{a_n + 1} \text{ for } n \geq 1 \\ a_1 = 1 \end{cases}.$$

Find $\lim_{n\to\infty} a_n$.

n	a_n
1	1
2	1.414213562
3	1.553773974
4	1.598053182
5	1.611847754
10	1.618016542
15	1.618033940

Solution

Suppose $\lim_{n\to\infty} a_n = a$. *Then* $\lim_{n\to\infty} a_{n+1} = a$ *and thus*

$$\begin{aligned} a &= \sqrt{a+1} \\ a^2 &= a+1 \\ a^2 - a - 1 &= 0 \end{aligned}$$

By solving the quadratic equation, we have

$$a = \frac{1+\sqrt{5}}{2} \text{ or } \frac{1-\sqrt{5}}{2}.$$

It is obvious that $a > 0$. *Therefore*

$$a = \frac{1+\sqrt{5}}{2} \approx 1.6180339887$$

Solution

The above solution is not complete. The solution is valid only after we have proved that $\lim_{n\to\infty} a_n$ *exists and is positive. This can be done by using* **monotone convergent theorem**. *We are going to show that* a_n *is* **bounded** *and* **monotonic**.

Boundedness

We prove that $1 \leq a_n < 2$ *for all* $n \geq 1$ *by induction.*
(Base case) When $n = 1$, *we have* $a_1 = 1$ *and* $1 \leq a_1 < 2$.
(Induction step) Assume that $1 \leq a_k < 2$. *Then*

$$\begin{aligned} a_{k+1} &= \sqrt{a_k + 1} \geq \sqrt{1+1} > 1 \\ a_{k+1} &= \sqrt{a_k + 1} < \sqrt{2+1} < 2 \end{aligned}$$

Thus $1 \leq a_n < 2$ *for any* $n \geq 1$ *which implies that* a_n *is bounded.*

Solution

Monotonicity

We prove that $a_{n+1} > a_n$ for any $n \geq 1$ by induction.
(Base case) When $n = 1$, $a_1 = 1$, $a_2 = \sqrt{2}$ and thus $a_2 > a_1$.
(Induction step) Assume that

$$a_{k+1} > a_k \text{ (Induction hypothesis).}$$

Then

$$\begin{aligned} a_{k+2} &= \sqrt{a_{k+1}+1} > \sqrt{a_k+1} \text{ (by induction hypothesis)} \\ &= a_{k+1} \end{aligned}$$

This completes the induction step and thus a_n is strictly increasing.
We have proved that a_n is bounded and strictly increasing. Therefore a_n is convergent by monotone convergence theorem. Since $a_n \geq 1$ for any n, we have $\lim_{n\to\infty} a_n \geq 1$ is positive.

Theorem

Let

$$a_n = \left(1+\frac{1}{n}\right)^n$$

$$b_n = \sum_{k=0}^{n} \frac{1}{k!} = 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}$$

Then

1. $a_n < b_n$ *for any* $n > 1$.
2. a_n *and* b_n *are convergent and*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$$

The limit of the two sequences is the important Euler's number

$$e \approx 2.71828\,18284\,59045\,23536\ldots.$$

which is also known as the Napier's constant.

Proof

Observe that by binomial theorem,

$$
\begin{aligned}
a_n &= \left(1+\frac{1}{n}\right)^n \\
&= 1 + n\cdot\frac{1}{n} + \frac{n(n-1)}{2!}\cdot\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\cdot\frac{1}{n^3} + \cdots + \frac{1}{n^n} \\
&= 1 + 1 + \frac{1}{2!}\cdot\frac{n-1}{n} + \frac{1}{3!}\cdot\frac{(n-1)(n-2)}{n^2} + \cdots + \frac{1}{n!}\cdot\frac{(n-1)\cdots 1}{n^{n-1}} \\
&= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots \\
&\quad + \frac{1}{n!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)
\end{aligned}
$$

Proof.

Boundedness: For any $n > 1$, we have

$$\begin{aligned}
a_n &= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \cdots \\
&\quad + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\cdots\left(1 - \frac{n-1}{n}\right) \\
&< 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} = b_n \\
&\leq 1 + 1 + \frac{1}{2^1} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} \\
&= 1 + 2\left(1 - \frac{1}{2^n}\right) \\
&< 3.
\end{aligned}$$

Thus $1 < a_n < b_n < 3$ for any $n > 1$. Therefore a_n and b_n are bounded. □

Proof

Monotonicity: *For any $n \geq 1$, we have*

$$\begin{aligned} a_n &= 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\cdots \\ &\quad +\frac{1}{n!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{n-1}{n}\right) \\ &< 1+1+\frac{1}{2!}\left(1-\frac{1}{n+1}\right)+\frac{1}{3!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)+\cdots \\ &\quad +\frac{1}{n!}\left(1-\frac{1}{n+1}\right)\cdots\left(1-\frac{n-1}{n+1}\right) \\ &\quad +\frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\cdots\left(1-\frac{n}{n+1}\right) \\ &= a_{n+1}. \end{aligned}$$

and it is obvious that $b_n < b_{n+1}$. Thus a_n are b_n are strictly increasing. Therefore a_n are b_n are convergent by monotone convergence theorem.

Proof

Alternative proof for monotonicity: *Recall that the arithmetic-geometric mean inequality says that for any positive real numbers $x_1, x_2, \ldots, x_k$, not all equal, we have*

$$x_1 x_2 \cdots x_k < \left(\frac{x_1 + x_2 + \cdots + x_k}{k}\right)^k.$$

Taking $k = n+1$, $x_1 = 1$ and $x_i = 1 + \frac{1}{n}$ for $i = 2, 3, \ldots, n+1$, we have

$$1 \cdot \left(1 + \frac{1}{n}\right)^n < \left(\frac{1 + n\left(1 + \frac{1}{n}\right)}{n+1}\right)^{n+1}$$

$$\left(1 + \frac{1}{n}\right)^n < \left(1 + \frac{1}{n+1}\right)^{n+1}.$$

Proof

Since $a_n < b_n$ for any $n > 1$, we have

$$\lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n.$$

On the other hand, for a fixed $m \ge 1$, define a sequence c_n (which depends on m) by

$$\begin{aligned} c_n &= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \cdots \\ &\quad + \frac{1}{m!}\left(1 - \frac{1}{n}\right)\cdots\left(1 - \frac{m-1}{n}\right) \end{aligned}$$

Proof

Then for any $n > m$, we have $a_n > c_n$ which implies that

$$\begin{aligned}
\lim_{n\to\infty} a_n &\geq \lim_{n\to\infty} c_n \\
&= 1 + 1 + \frac{1}{2!}\lim_{n\to\infty}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\lim_{n\to\infty}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots \\
&\quad + \frac{1}{m!}\lim_{n\to\infty}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{m-1}{n}\right) \\
&= 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{m!} \\
&= b_m.
\end{aligned}$$

Observe that m is arbitrary and thus

$$\lim_{n\to\infty} a_n \geq \lim_{m\to\infty} b_m = \lim_{n\to\infty} b_n.$$

Therefore

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n.$$

Example

Let $a_n = \dfrac{F_{n+1}}{F_n}$ where F_n is the Fibonacci's sequence defined by

$$\begin{cases} F_{n+2} = F_{n+1} + F_n \\ F_1 = F_2 = 1 \end{cases}.$$

Find $\lim_{n\to\infty} a_n$.

n	a_n
1	1
2	2
3	1.5
4	1.666666666
5	1.6
10	1.618181818
15	1.618032787
20	1.618033999

Theorem

For any $n \geq 1$,

1. $F_{n+2}F_n - F_{n+1}^2 = (-1)^{n+1}$
2. $F_{n+3}F_n - F_{n+2}F_{n+1} = (-1)^{n+1}$

Proof

1. *When* $n = 1$, *we have* $F_3F_1 - F_2^2 = 2 \cdot 1 - 1^2 = 1 = (-1)^2$. *Assume*

$$F_{k+2}F_k - F_{k+1}^2 = (-1)^{k+1}.$$

Then

$$\begin{aligned} F_{k+3}F_{k+1} - F_{k+2}^2 &= (F_{k+2} + F_{k+1})F_{k+1} - F_{k+2}^2 \\ &= F_{k+2}(F_{k+1} - F_{k+2}) + F_{k+1}^2 \\ &= -F_{k+2}F_k + F_{k+1}^2 \\ &= (-1)^{k+2} \text{ (by induction hypothesis)} \end{aligned}$$

Therefore $F_{n+2}F_n - F_{n+1}^2 = (-1)^{n+1}$ *for any* $n \geq 1$.

Proof.

The proof for the second statement is basically the same. When $n = 1$, we have $F_4F_1 - F_3F_2 = 3 \cdot 1 - 2 \cdot 1 = 1 = (-1)^2$. Assume

$$F_{k+3}F_k - F_{k+2}F_{k+1} = (-1)^{k+1}.$$

Then

$$\begin{aligned} F_{k+4}F_{k+1} - F_{k+3}F_{k+2} &= (F_{k+3} + F_{k+2})F_{k+1} - F_{k+3}F_{k+2} \\ &= F_{k+3}(F_{k+1} - F_{k+2}) + F_{k+2}F_{k+1} \\ &= -F_{k+3}F_k + F_{k+2}F_{k+1} \\ &= -(-1)^{k+1} \text{ (by induction hypothesis)} \\ &= (-1)^{k+2} \end{aligned}$$

Therefore $F_{n+3}F_n - F_{n+2}F_{n+1} = (-1)^{n+1}$ for any $n \geq 1$. □

Theorem

Let $a_n = \dfrac{F_{n+1}}{F_n}$.

1. *The sequence* $a_1, a_3, a_5, a_7, \cdots,$ *is strictly increasing.*
2. *The sequence* $a_2, a_4, a_6, a_8, \cdots,$ *is strictly decreasing.*

Proof.

For any $k \geq 1$, we have

$$\begin{aligned} a_{2k+1} - a_{2k-1} &= \frac{F_{2k+2}}{F_{2k+1}} - \frac{F_{2k}}{F_{2k-1}} = \frac{F_{2k+2}F_{2k-1} - F_{2k+1}F_{2k}}{F_{2k+1}F_{2k-1}} \\ &= \frac{(-1)^{2k}}{F_{2k+1}F_{2k-1}} = \frac{1}{F_{2k+1}F_{2k-1}} > 0 \end{aligned}$$

Therefore $a_1, a_3, a_5, a_7, \cdots,$ is strictly increasing. The second statement can be proved in a similar way. □

Theorem

$$\lim_{k\to\infty}(a_{2k+1}-a_{2k})=0$$

Proof.

For any $k\geq 1$,

$$\begin{aligned} a_{2k+1}-a_{2k} &= \frac{F_{2k+2}}{F_{2k+1}}-\frac{F_{2k+1}}{F_{2k}} \\ &= \frac{F_{2k+2}F_{2k}-F_{2k+1}^2}{F_{2k+1}F_{2k}}=\frac{1}{F_{2k+1}F_{2k}} \end{aligned}$$

Therefore

$$\lim_{k\to\infty}(a_{2k+1}-a_{2k})=\lim_{k\to\infty}\frac{1}{F_{2k+1}F_{2k}}=0.$$

$\square$

Theorem

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}$$

Proof

First we prove that $a_n = \frac{F_{n+1}}{F_n}$ *is convergent.*
a_n *is bounded.* $(1 \leq a_n \leq 2$ *for any* $n.)$
a_{2k+1} *and* a_{2k} *are convergent. (They are bounded and monotonic.)*

$$\lim_{k\to\infty}(a_{2k+1} - a_{2k}) = 0 \Rightarrow \lim_{k\to\infty} a_{2k+1} = \lim_{k\to\infty} a_{2k}$$

It follows that a_n *is convergent and*

$$\lim_{n\to\infty} a_n = \lim_{k\to\infty} a_{2k+1} = \lim_{k\to\infty} a_{2k}.$$

Proof.

To evaluate the limit, suppose $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = L$. Then

$$L = \lim_{n\to\infty} \frac{F_{n+2}}{F_{n+1}} = \lim_{n\to\infty} \frac{F_{n+1} + F_n}{F_{n+1}} = \lim_{n\to\infty} \left(1 + \frac{F_n}{F_{n+1}}\right) = 1 + \frac{1}{L}$$

$$L^2 - L - 1 = 0$$

By solving the quadratic equation, we have

$$L = \frac{1+\sqrt{5}}{2} \text{ or } \frac{1-\sqrt{5}}{2}.$$

We must have $L \geq 1$ since $a_n \geq 1$ for any n. Therefore

$$L = \frac{1+\sqrt{5}}{2}.$$

□

Remarks

The limit can be calculate directly using the formula

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

where

$$\alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2}$$

are the roots of the quadratic equation

$$x^2 - x - 1 = 0.$$

Definition (Convergence of infinite series)

We say that an **infinite series**

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots$$

is **convergent** if the sequence of partial sums $s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n$ is convergent. If the infinite series is convergent, then we define

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k.$$

Definition (Absolute convergence)

We say that an infinite series $\sum_{k=1}^{\infty} a_k$ is **absolutely convergent** if $\sum_{k=1}^{\infty} |a_k|$ is convergent.

Example

Series	Convergency	Absolute convergency
$\sum_{k=0}^{\infty} \frac{1}{2^k} = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$	2	Yes
$\sum_{k=0}^{\infty} \frac{1}{k!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$	e	Yes
$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$	divergent	No
$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$	$\frac{\pi^2}{6}$	Yes
$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$	$\ln 2$	No
$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$	$\frac{\pi}{4}$	No

Theorem

If $\sum_{k=1}^{\infty} a_k$ *is convergent, then* $\lim_{k\to\infty} a_k = 0$.

The converse is not true. $\lim_{k\to\infty} \frac{1}{k} = 0$ but $\sum_{k=1}^{\infty} \frac{1}{k}$ is divergent.

Theorem

If $\sum_{k=1}^{\infty} |a_k|$ *is convergent, then* $\sum_{k=1}^{\infty} a_k$ *is convergent.*

Absolutely convergent $\Rightarrow$ **Convergent**

The converse is not true. $\lim_{k\to\infty} \frac{(-1)^{k+1}}{k}$ is convergent but $\sum_{k=1}^{\infty} \frac{1}{k}$ is divergent.

Theorem (Comparison test for convergence)

If $0 \leq |a_k| \leq b_k$ for any k and $\sum_{k=0}^{\infty} b_k$ is convergent. Then $\sum_{k=0}^{\infty} a_k$ is convergent.

Theorem (Alternating series test)

If $a_0 > a_1 > a_2 > \cdots > 0$ is a decreasing sequence of positive real numbers and $\lim_{k\to\infty} a_k = 0$, then $\sum_{k=0}^{\infty} (-1)^k a_k$ is convergent.

Definition (Exponential function)

The **exponential function** is defined for real number $x \in \mathbb{R}$ by

$$\begin{aligned} e^x &= \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \end{aligned}$$

1. It can be proved that the two limits in the definition exist and converge to the same value for any real number x.
2. e^x is just a notation for the exponential function. One should not interpret it as 'e to the power x'.

Theorem

For any $x, y \in \mathbb{R}$, *we have*

$$e^{x+y} = e^x e^y.$$

Caution! One cannot use law of indices to prove the above identity. It is because e^x is just a notation for the exponential function and it does not mean 'e to the power x'. In fact we have not defined what a^x means when x is a real number which is not rational.

Proof.

$$
\begin{aligned}
e^{x+y} &= \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!} \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} \cdot \frac{x^m y^{n-m}}{n!} \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{x^m y^{n-m}}{m!(n-m)!} \\
&= \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{x^m y^k}{m!k!} \\
&= \sum_{m=0}^{\infty} \frac{x^m}{m!} \sum_{k=0}^{\infty} \frac{y^k}{k!} \\
&= e^x e^y
\end{aligned}
$$

Here we have changed the order of summation in the 4th equality. We can do this because the series for exponential function is absolutely convergent. □

Theorem

1. $e^x > 0$ *for any real number* x.
2. e^x *is strictly increasing.*

Proof.

1. For any $x > 0$, we have $e^x > 1 + x > 1$. If $x < 0$, then

$$\begin{aligned} e^x e^{-x} &= e^{x+(-x)} = e^0 = 1 \\ e^x &= \frac{1}{e^{-x}} > 0 \end{aligned}$$

   since $e^{-x} > 1$. Therefore $e^x > 0$ for any $x \in \mathbb{R}$.

2. Let x, y be real numbers with $x < y$. Then $y - x > 0$ which implies $e^{y-x} > 1$. Therefore

$$e^y = e^{x+(y-x)} = e^x e^{y-x} > e^x.$$

□

Definition (Logarithmic function)

The **logarithmic function** is the function $\ln : \mathbb{R}^+ \to \mathbb{R}$ defined for $x > 0$ by

$$y = \ln x \text{ if } e^y = x.$$

In other words, $\ln x$ is the inverse function of e^x.

It can be proved that for any $x > 0$, there exists unique real number y such that $e^y = x$.

Theorem

1. $\ln xy = \ln x + \ln y$
2. $\ln \frac{x}{y} = \ln x - \ln y$
3. $\ln x^n = n \ln x$ *for any integer* $n \in \mathbb{Z}$.

Proof.

1. Let $u = \ln x$ and $v = \ln y$. Then $x = e^u$, $y = e^v$ and we have

$$xy = e^u e^v = e^{u+v} = e^{\ln x + \ln y}$$

which means $\ln xy = \ln x + \ln y$.

Other parts can be proved similarly. □

Definition (Cosine and sine functions)

The **cosine** and **sine** functions are defined for real number $x \in \mathbb{R}$ by the infinite series

$$\begin{aligned} \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \end{aligned}$$

1. When the sine and cosine are interpreted as trigonometric ratios, the angles are measured in radian. $(180^0 = \pi)$
2. The series for cosine and sine are convergent for any real number $x \in \mathbb{R}$.

There are four more trigonometric functions namely tangent, cotangent, secant and cosecant functions. All of them are defined in terms of sine and cosine.

Definition (Trigonometric functions)

$$\begin{aligned}
\tan x &= \frac{\sin x}{\cos x}, \text{ for } x \neq \frac{2k+1}{2}\pi, \ k \in \mathbb{Z} \\
\cot x &= \frac{\cos x}{\sin x}, \text{ for } x \neq k\pi, \ k \in \mathbb{Z} \\
\sec x &= \frac{1}{\cos x}, \text{ for } x \neq \frac{2k+1}{2}\pi, \ k \in \mathbb{Z} \\
\csc x &= \frac{1}{\sin x}, \text{ for } x \neq k\pi, \ k \in \mathbb{Z}
\end{aligned}$$

Theorem (Trigonometric identities)

1. $\cos^2 x + \sin^2 x = 1; \quad \sec^2 x - \tan^2 x = 1; \quad \csc^2 x - \cot^2 x = 1$
2. $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y;$
   $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y;$
   $\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$
3. $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x;$
   $\sin 2x = 2 \sin x \cos x;$
   $\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}$
4. $2 \cos x \cos y = \cos(x + y) + \cos(x - y)$
   $2 \cos x \sin y = \sin(x + y) - \sin(x - y)$
   $2 \sin x \sin y = \cos(x - y) - \cos(x + y)$
5. $\cos x + \cos y = 2 \cos\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)$
   $\cos x - \cos y = -2 \sin\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right)$
   $\sin x + \sin y = 2 \sin\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)$
   $\sin x - \sin y = 2 \cos\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right)$

Definition (Hyperbolic function)

The **hyperbolic functions** are defined for $x \in \mathbb{R}$ by

$$\begin{aligned} \cosh x &= \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots \\ \sinh x &= \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots \end{aligned}$$

Theorem (Hyperbolic identities)

1. $\cosh^2 x - \sinh^2 x = 1$
2. $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$
   $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$
3. $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x;$
   $\sinh 2x = 2 \sinh x \cosh x$

Definition (Limit of function)

Let $f(x)$ be a real valued function.

1. We say that a real number L is a limit of $f(x)$ at $x = a$ if for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\text{if } 0 < |x - a| < \delta, \text{ then } |f(x) - L| < \epsilon$$

and write

$$\lim_{x \to a} f(x) = L.$$

2. We say that a real number L is a limit of $f(x)$ at $+\infty$ if for any $\epsilon > 0$, there exists $R > 0$ such that

$$\text{if } x > R, \text{ then } |f(x) - L| < \epsilon$$

and write

$$\lim_{x \to +\infty} f(x) = L.$$

The limit of $f(x)$ at $-\infty$ is defined similarly.

1. Note that for the limit of $f(x)$ at $x = a$ to exist, $f(x)$ may not be defined at $x = a$ and even if $f(a)$ is defined, the value of $f(a)$ does not affect the value of the limit at $x = a$.
2. The limit of $f(x)$ at $x = a$ may not exists. However the limit is unique if it exists.

Theorem (Limit of function and limit of sequence)

Let $f(x)$ be a real valued function. Then

$$\lim_{x\to a} f(x) = L$$

if and only if for any sequence x_n with $\lim_{n\to\infty} x_n = a$, we have

$$\lim_{n\to\infty} f(x_n) = L.$$

Theorem

Let $f(x)$, $g(x)$ be functions and c be a real number. Then

1. $\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x)$
3. $\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$
4. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ *if* $\lim_{x\to a} g(x) \neq 0$.

Theorem

Let $f(u)$ be a function of u and $u = g(x)$ is a function of x. Suppose

1. $\lim_{x \to a} g(x) = b \in [-\infty, +\infty]$
2. $\lim_{u \to b} f(u) = L$
3. $g(x) \neq b$ *when* $x \neq a$ *or* $f(b) = L$.

Then

$$\lim_{x \to a} f \circ g(x) = L.$$

Theorem (Squeeze theorem)

Let $f(x), g(x), h(x)$ be real valued functions. Suppose

1. $f(x) \leq g(x) \leq h(x)$ *for any* $x \neq a$ *on a neighborhood of* a, *and*
2. $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$.

Then the limit of $g(x)$ at $x = a$ exists and

$$\lim_{x \to a} g(x) = L.$$

Theorem

Suppose $f(x)$ is bounded and $\lim_{x \to a} g(x) = 0$. Then

$$\lim_{x \to a} f(x)g(x) = 0.$$

Theorem

1. $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$
2. $\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$
3. $\lim_{x\to 0} \frac{\sin x}{x} = 1$

Proof. $\displaystyle\lim_{x\to 0}\frac{e^x-1}{x}=1.$

For any $-1<x<1$ with $x\neq 0$, we have

$$\begin{aligned}\frac{e^x-1}{x} &= 1+\frac{x}{2!}+\frac{x^2}{3!}+\frac{x^3}{4!}+\frac{x^4}{5!}+\cdots\\ &\leq 1+\frac{x}{2}+\left(\frac{x^2}{4}+\frac{x^2}{8}+\frac{x^2}{16}+\cdots\right)=1+\frac{x}{2}+\frac{x^2}{2}\\ \frac{e^x-1}{x} &= 1+\frac{x}{2!}+\frac{x^2}{3!}+\frac{x^3}{4!}+\cdots\\ &\geq 1+\frac{x}{2}-\left(\frac{x^2}{4}+\frac{x^2}{8}+\frac{x^2}{16}+\cdots\right)=1+\frac{x}{2}-\frac{x^2}{2}\end{aligned}$$

and $\displaystyle\lim_{x\to 0}(1+\frac{x}{2}+\frac{x^2}{2})=\lim_{x\to 0}(1+\frac{x}{2}-\frac{x^2}{2})=1$. Therefore $\displaystyle\lim_{x\to 0}\frac{e^x-1}{x}=1$. □

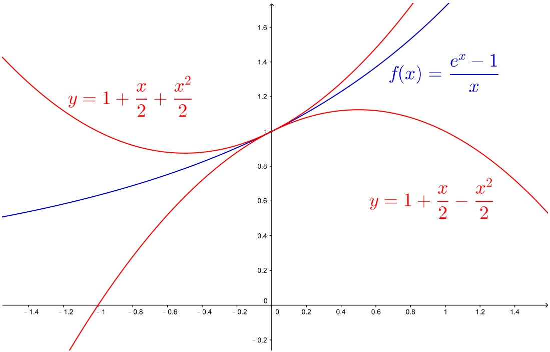

Figure: $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

Proof. $\lim_{x\to 0}\frac{\ln(1+x)}{x}=1.$

Let $y=\ln(1+x)$. Then

$$\begin{aligned} e^y &= 1+x \\ x &= e^y-1 \end{aligned}$$

and $x\to 0$ as $y\to 0$. We have

$$\begin{aligned} \lim_{x\to 0}\frac{\ln(1+x)}{x} &= \lim_{y\to 0}\frac{y}{e^y-1} \\ &= 1 \end{aligned}$$

□

Note that the first part implies $\lim_{y\to 0}(e^y-1)=0$.

Proof. $\lim_{x\to 0}\frac{\sin x}{x}=1.$

Note that
$$\frac{\sin x}{x}=1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\frac{x^8}{9!}-\frac{x^{10}}{11!}+\cdots.$$
For any $-1<x<1$ with $x\neq 0$, we have
$$\begin{aligned}\frac{\sin x}{x} &= 1-\left(\frac{x^2}{3!}-\frac{x^4}{5!}\right)-\left(\frac{x^6}{7!}-\frac{x^8}{9!}\right)-\cdots\leq 1\\ \frac{\sin x}{x} &= 1-\frac{x^2}{6}+\left(\frac{x^4}{5!}-\frac{x^6}{7!}\right)+\left(\frac{x^8}{9!}-\frac{x^{10}}{11!}\right)+\cdots\geq 1-\frac{x^2}{6}\end{aligned}$$
and $\lim_{x\to 0}1=\lim_{x\to 0}(1-\frac{x^2}{6})=1$. Therefore
$$\lim_{x\to 0}\frac{\sin x}{x}=1.$$
□

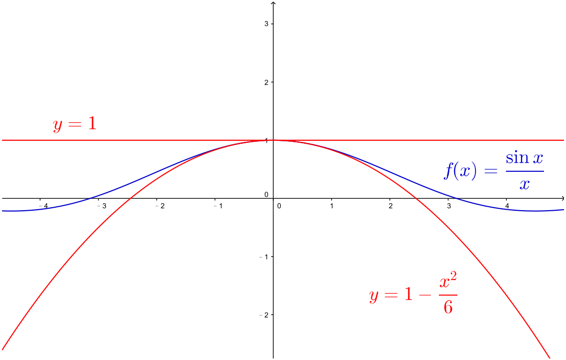

Figure: $\lim_{x \to 0} \frac{\sin x}{x} = 1$

Theorem

Let k be a positive integer.

1. $\displaystyle\lim_{x\to+\infty}\frac{x^k}{e^x}=0$
2. $\displaystyle\lim_{x\to+\infty}\frac{(\ln x)^k}{x}=0$

Proof.

1. For any $x > 0$,
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots > \frac{x^{k+1}}{(k+1)!}$$
and thus
$$0 < \frac{x^k}{e^x} < \frac{(k+1)!}{x}.$$
Moreover $\lim_{x\to\infty} \frac{(k+1)!}{x} = 0$. Therefore
$$\lim_{x\to+\infty} \frac{x^k}{e^x} = 0.$$
2. Let $x = e^y$. Then $x \to +\infty$ as $y \to +\infty$ and $\ln x = y$. We have
$$\lim_{x\to+\infty} \frac{(\ln x)^k}{x} = \lim_{y\to+\infty} \frac{y^k}{e^y} = 0.$$

□

Example

1. $$\lim_{x\to 4}\frac{x^2-16}{\sqrt{x}-2} = \lim_{x\to 4}\frac{(x-4)(x+4)(\sqrt{x}+2)}{(\sqrt{x}-2)(\sqrt{x}+2)} = \lim_{x\to 4}\frac{(x-4)(x+4)(\sqrt{x}+2)}{x-4} = \lim_{x\to 4}(x+4)(\sqrt{x}+2) = 32$$

2. $$\lim_{x\to+\infty}\frac{3e^{2x}+e^x-x^4}{4e^{2x}-5e^x+2x^4} = \lim_{x\to+\infty}\frac{3+e^{-x}-x^4e^{-2x}}{4-5e^{-x}+2x^4e^{-2x}} = \frac{3}{4}$$

3. $$\lim_{x\to+\infty}\frac{\ln(2e^{4x}+x^3)}{\ln(3e^{2x}+4x^5)} = \lim_{x\to+\infty}\frac{4x+\ln(2+x^3e^{-4x})}{2x+\ln(3+4x^5e^{-2x})} = \lim_{x\to+\infty}\frac{4+\frac{\ln(2+x^3e^{-4x})}{x}}{2+\frac{\ln(3+4x^5e^{-2x})}{x}} = 2$$

4. $$\lim_{x\to-\infty}(x+\sqrt{x^2-2x}) = \lim_{x\to-\infty}\frac{(x+\sqrt{x^2-2x})(x-\sqrt{x^2-2x})}{x-\sqrt{x^2-2x}} = \lim_{x\to-\infty}\frac{2x}{x-\sqrt{x^2-2x}} = \lim_{x\to-\infty}\frac{2}{1+\sqrt{1-\frac{2}{x}}} = 1$$

Example

5. $$\lim_{x\to 0}\frac{\sin 6x-\sin x}{\sin 4x-\sin 3x} = \lim_{x\to 0}\frac{\frac{6\sin 6x}{6x}-\frac{\sin x}{x}}{\frac{4\sin 4x}{4x}-\frac{3\sin 3x}{3x}} = \frac{6-1}{4-3} = 5$$

6. $$\begin{aligned}\lim_{x\to 0}\frac{1-\cos x}{x\tan x} &= \lim_{x\to 0}\frac{(1-\cos x)(1+\cos x)}{x\frac{\sin x}{\cos x}(1+\cos x)}\\ &= \lim_{x\to 0}\frac{(1-\cos^2 x)\cos x}{x\sin x(1+\cos x)}\\ &= \lim_{x\to 0}\left(\frac{\sin x}{x}\right)\frac{\cos x}{1+\cos x} = \frac{1}{2}\end{aligned}$$

7. $$\lim_{x\to 0}\frac{e^{2x}-1}{\ln(1+3x)} = \lim_{x\to 0}\frac{2}{3}\cdot\frac{e^{2x}-1}{2x}\cdot\frac{3x}{\ln(1+3x)} = \frac{2}{3}$$

8. $$\begin{aligned}\lim_{x\to 0}\frac{x\ln(1+\sin x)}{1-\sqrt{\cos x}} &= \lim_{x\to 0}\frac{x(1+\sqrt{\cos x})(1+\cos x)\ln(1+\sin x)}{1-\cos^2 x}\\ &= \lim_{x\to 0}\frac{x}{\sin x}\cdot\frac{\ln(1+\sin x)}{\sin x}(1+\sqrt{\cos x})(1+\cos x)\\ &= 4\end{aligned}$$

Definition (Continuity)

Let $f(x)$ be a real valued function. We say that $f(x)$ is **continuous** at $x = a$ if

$$\lim_{x \to a} f(x) = f(a).$$

In other words, $f(x)$ is continuous at $x = a$ if for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\text{if } |x - a| < \delta, \text{ then } |f(x) - f(a)| < \epsilon.$$

We say that $f(x)$ is continuous on an interval in $\mathbb{R}$ if $f(x)$ is continuous at every point on the interval.

Theorem

Let $f(u)$ and $u = g(x)$ be functions. Suppose $f(u)$ is continuous and the limit of $g(x)$ at $x = a$ exists. Then

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right).$$

Theorem

1. *For any non-negative integer* n, $f(x) = x^n$ *is continuous on* $\mathbb{R}$.
2. *The functions* $e^x, \cos x, \sin x$ *are continuous on* $\mathbb{R}$.
3. *The logarithmic function* $\ln x$ *is continuous on* $\mathbb{R}^+$.

Proof.

We prove the continuity of x^n and e^x.
(Continuity of x^n)

$$\lim_{x\to a} x = a \Rightarrow \lim_{x\to a} x^n = a^n.$$

Thus x^n is continuous at $x = a$ for any real number a.
(Continuity of e^x)

$$\begin{aligned}\lim_{x\to a} e^x &= \lim_{h\to 0} e^{a+h}\\ &= \lim_{h\to 0} e^a e^h\\ &= e^a\end{aligned}$$

Thus e^x is continuous at $x = a$ for any real number a. □

Theorem

Suppose $f(x)$, $g(x)$ are continuous functions and c is a real number. Then the following functions are continuous.

1. $f(x) + g(x)$
2. $cf(x)$
3. $f(x)g(x)$
4. $\dfrac{f(x)}{g(x)}$ *at the points where* $g(x) \neq 0$.
5. $f \circ g(x)$

Theorem

A function $f(x)$ is continuous at $x = a$ if

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = f(a).$$

The theorem is usually used to check whether a piecewise defined function is continuous.

Example

Given that the function

$$f(x) = \begin{cases} 2x - 1 & \text{if } x < 2 \\ a & \text{if } x = 2 \\ x^2 + b & \text{if } x > 2 \end{cases}$$

is continuous at $x = 2$. Find the value of a and b.

Solution

Note that

$$\begin{aligned} \lim_{x \to 2^-} f(x) &= \lim_{x \to 2^-} (2x - 1) = 3 \\ \lim_{x \to 2^+} f(x) &= \lim_{x \to 2^+} (x^2 + b) = 4 + b \\ f(2) &= a \end{aligned}$$

Since $f(x)$ is continuous at $x = 2$, we have $3 = 4 + b = a$ which implies $a = 3$ and $b = -1$.

Definition (Intervals)

Let $a < b$ be real numbers. We define the intervals

$$
\begin{aligned}
(a, b) &= \{x \in \mathbb{R} : a < x < b\} \\
[a, b] &= \{x \in \mathbb{R} : a \leq x \leq b\} \\
(a, b] &= \{x \in \mathbb{R} : a < x \leq b\} \\
[a, b) &= \{x \in \mathbb{R} : a \leq x < b\} \\
(a, +\infty) &= \{x \in \mathbb{R} : a < x\} \\
[a, +\infty) &= \{x \in \mathbb{R} : a \leq x\} \\
(-\infty, b) &= \{x \in \mathbb{R} : x < b\} \\
(-\infty, b] &= \{x \in \mathbb{R} : x \leq b\} \\
(-\infty, +\infty) &= \mathbb{R}
\end{aligned}
$$

Definition (Open, closed and bounded sets)

Let $D \subset \mathbb{R}$ be a subset of $\mathbb{R}$.

1. We say that D is **open** if for any $x \in D$, there exits $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset D$.
2. We say that D is **closed** if for any sequence $x_n \in D$ of numbers in D which converges to $x \in \mathbb{R}$, we have $x \in D$.
3. We say that D is **bounded** if there exists real number M such that for any $x \in D$, we have $|x| < M$.

Note that a subset $D \subset \mathbb{R}$ is open if and only if its complement $D^c = \{x \in \mathbb{R} : x \notin D\}$ in $\mathbb{R}$ is closed.

Example

Let $a < b$ be real numbers.

Subset	open	closed	bounded
$\emptyset$	Yes	Yes	Yes
(a, b)	Yes	No	Yes
$[a, b]$	No	Yes	Yes
$(a, b]$,$[a, b)$	No	No	Yes
$(a, +\infty)$,$(-\infty, b)$	Yes	No	No
$[a, +\infty)$,$(-\infty, b]$	No	Yes	No
$(-\infty, +\infty)$	Yes	Yes	No
$(-\infty, a) \cup [b, +\infty)$	No	No	No

Theorem (Intermediate value theorem)

Suppose $f(x)$ is a function which is **continuous** *on a* **closed and bounded** *interval $[a, b]$. Then for any real number η between $f(a)$ and $f(b)$, there exists $\xi \in (a, b)$ such that $f(\xi) = \eta$.*

Theorem (Extreme value theorem)

Suppose $f(x)$ is a function which is **continuous** *on a* **closed and bounded** *interval $[a, b]$. Then there exists $\alpha, \beta \in [a, b]$ such that for any $x \in [a, b]$, we have*

$$f(\alpha) \leq f(x) \leq f(\beta).$$