Lecture 9

Goal:

prop (*): Let $g \in G^{occ}_{\geq 0}$. Then $\exists!$ $u \in U^-_{\geq 0}$, $u' \in U^+_{\geq 0}$, $t \in T_{>0}$ s.t. $gu = uu't$.

Last time:

Perron's thm: Let A be a positive matrix. Then $\exists!$ eigenvalue λ s.t. the corresponding eigenvector is positive and λ is the unique eigenvalue with the largest absolute value and the unique eigenvalue with positive eigenvector

First we generalize Perron's thm to nonnegative matrices

Def. A matrix is nonnegative if all entries ≥ 0. A matrix is decomposable if it is block diagonal after some permutation: $\exists$ a nontrivial partition $\{1, \dots, n\} = I_1 \sqcup I_2$ s.t. $a_{ij} = a_{ji} = 0$ $\forall\, i \in I_1, j \in I_2$

Example: $\begin{pmatrix} * & & * & \\ & * & & * \\ * & & * & \\ & * & & * \end{pmatrix}$ is decomposable (let $I_1 = \{1,3\}$, $I_2 = \{2,4\}$)

lemma. Let A be a nonnegative matrix. Then A is indecomposable iff A^m is positive for some $m \in \mathbb{N}$

proof (sketch): We draw a graph with vertices in $\{1, \dots, n\}$ and $i \to j$ if $a_{ij} > 0$ $\Rightarrow$

indecomposable iff the graph is connected iff A^m is positive for some m.

Remark: We may take $m=n$.

Perron-Frobenius thm. Let A be a nonnegative matrix s.t. A^m is positive. Then $\exists!$ eigenvalue λ s.t. the corresponding eigenvector is positive and λ is the unique eigenvalue with the largest absolute value and the unique eigenvalue with positive eigenvector

proof: Let λ be an eigenvalue of A with $|\lambda| = \rho(A)$ (largest possible absolute value), v a corresponding eigenvector $\Rightarrow Av = \lambda v \Rightarrow A^m v = \lambda^m v$. But $|\lambda^m| = \rho(A)^m = \rho(A^m)$. By Perron's thm, λ^m is the unique eigenvalue with absolute value $\rho(A^m)$. So all the other eigenvalues λ' of A has $|\lambda'| < \rho(A) = |\lambda|$. Up to scalar, v is a positive eigenvector, as

$$\underset{\text{positive}}{\underset{\uparrow}{A}}v = \lambda \underset{\text{positive}}{\underset{\uparrow}{v}} \Rightarrow \lambda \text{ is positive.}$$

$\boxtimes$

Quick Review of the flag variety:

Bruhat decomposition. $G = \bigsqcup_{w \in W} B^+ w B^+$

The flag variety $\mathcal{B} = G/B^+$. We have the decomposition into the Schubert cells. $\mathcal{B} = \bigsqcup_{w \in W} B^+ w B^+ / B^+$.

Geometric fact: $\mathcal{B}$ is a proper variety (Springer's book on LAG. 6.2)

Representation fact: Let V be a faithful repr of G. Then $B \hookrightarrow \mathbb{P}(V)$ is a closed embedding.

Example: $G = GL_n$ $Gr(k,n)$ is the space of all k-dim subspaces of $\mathbb{C}^n$

$Gr(1,n)$ — lines in $\mathbb{C}^n$

$\approx \mathbb{P}(V)$. V is the tautological repr of G.

$Gr(2,n)$ — 2-dim

Plucker embedding $\downarrow$

$\mathbb{P}(\Lambda^2\mathbb{C}^n)$ 2-dim subspaces of $\mathbb{C}^n \longrightarrow$ lines in $\Lambda^2\mathbb{C}^n$

$\begin{bmatrix} a_1 & \cdots & a_n \\ b_1 & \cdots & b_n \end{bmatrix} \longmapsto \det\left(\begin{bmatrix} a_i & a_j \\ b_i & b_j \end{bmatrix}\right)_{1 \le i < j \le n}$

The image of $Gr(2,n)$ in $\mathbb{P}(\Lambda^2\mathbb{C}^n)$ is given by the Plucker relation. In particular, $Gr(2,n)$ is closed in $\mathbb{P}(\Lambda^2\mathbb{C}^n)$.

In general, $Gr(k,n) \hookrightarrow \mathbb{P}(\Lambda^k\mathbb{C}^n)$ closed embedding

The flag variety:

Fact: $\mathcal{B} = \{ 0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = \mathbb{C}^n : \dim V_i = i \}$.

1. G acts transitively on $\mathcal{B}$: Any flag is determined by a basis on $\mathbb{C}^n$ and G acts transitively on the set of basis.
2. The standard flag $(0 = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n)$ where $V_i = \mathrm{span}(e_1, \cdots, e_i)$

   The isotropy group is $\begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ & & * \end{pmatrix} = B^+$

Now $\mathcal{B} = \{(V_1, \cdots, V_{n-1}) \in Gr(1,n) \times \cdots \overset{\text{proper}}{\times} Gr(n-1,n) ; \ V_i \overset{\text{closed}}{\subset} V_{i+1} \ \forall i\} \Rightarrow \mathcal{B}$ is a closed subvariety of $Gr(1,n) \times \cdots \times Gr(n-1,n) \subseteq \mathbb{P}(\Lambda^1 \mathbb{C}^n) \times \cdots \times \mathbb{P}(\Lambda^{n-1}\mathbb{C}^n) \subseteq \mathbb{P}(\Lambda^1\mathbb{C}^n \otimes \cdots \Lambda^{n-1}\mathbb{C}^n)$

Now $\mathcal{B} \hookrightarrow \mathbb{P}(V)$ closed embedding $\quad B \mapsto L_B$ (line stabilized by B)

B: Borel subgroup

G-action: $\quad g \cdot B = gBg^{-1} \mapsto L_{gBg^{-1}} = g \cdot L_B$ ($\mathcal{B}$: the variety of all the Borel subgroups of G as any two Borel are conj in G, and $N_G(B) = B$)

Step 1: For simply laced group G. $\exists!\ u \in U^-_{>0}$ s.t. $g \in u \cdot B^+ = uB^+u^{-1}$

For simply laced G, we may use canonical basis, we have $\rho: G \to GL(V)$ (matrices w.r.t. canonical basis) $G_{>0} \to$ positive matrices

Let $\mathcal{B}_{>0} = \{u \cdot B^+ : u \in U^-_{>0}\} \subseteq \mathcal{B}$ and $\mathcal{B}_{\geq 0}$ be its closure in $\mathcal{B}$ (in the Hausdorff topology)

Remark: The Hausdorff closure of $\mathbb{R}_{>0}$ in $\mathbb{C}$ is $\mathbb{R}_{\geq 0}$.

The Zariski closure of $\mathbb{R}_{>0}$ in $\mathbb{C}$ is $\mathbb{C}$.

Let $\mathbb{P}(V)_{>0}$ ($\mathbb{P}(V)_{\geq 0}$) be the projective line of positive (nonnegative) vectors in V.

$L_{B^+} = (1, 0, \cdots, 0) \in \mathbb{P}(V)_{\geq 0}$

highest weight vector

$L_{u \cdot B^+} = u \cdot (1, 0, \cdots, 0) \in \mathbb{P}(V)_{>0} \quad \forall u \in U^-_{>0}$

So $\mathcal{B}_{>0} \subset \mathbb{P}(V)_{>0} \qquad \mathcal{B}_{\geq 0} \subset \mathbb{P}(V)_{\geq 0}$

$G_{>0} = U^-_{>0} T_{>0} U^+_{>0}$ and $G_{>0} U^-_{>0} \subseteq G_{>0} \subseteq U^-_{>0} B^+$

use the relation on the TP

So $G_{>0}$ action on $\mathcal{B}$ stabilizes $\mathcal{B}_{>0} = U^-_{>0} \cdot B^+$, hence stabilizes $\mathcal{B}_{\geq 0}$

Also. $G_{\geq 0}$ stabilizes $\mathcal{B}_{>0}$ and $\mathcal{B}_{\geq 0}$

Let $g \in G^{osc}_{\geq 0}$. By Perron's thm, $\exists!$ positive eigenvector v, $V = L_g \oplus V'$, where $L_g = \mathbb{C}v$.

$V' = \oplus$ other generalized eigenspaces

Remark: $g^m > 0$, in fact we could use the Perron-Frobenius thm

lemma. Any g-stable closed subspace X of $\mathbb{P}(V)_{\geq 0}$ contains L.

proof: Let $v' \in V$ s.t. $\mathbb{R} \cdot v' \in X$. $v' = av + v''$, where $a \in \mathbb{C}$, $a \neq 0$, $v'' \in V' \Rightarrow \lim_{l \to \infty} g^l \cdot v' / \lambda^l = av$, as λ is the unique eigenvalue with largest absolute norm. But X is closed $\Rightarrow L \in X$. $\boxtimes$

Applying the lemma to $\mathcal{B}_{\geq 0}$, we have $L_g \in \mathcal{B}_{\geq 0}$, but L_g has all the entries > 0 $\Rightarrow$

$L_g = [1, \cdots, *]$ $\Rightarrow L_g \in U^- \cdot B^+$. Since $U^-_{\geq 0}$ is closed in $U^- \Rightarrow L_g \in U^-_{\geq 0} \cdot B^+$.

highest weight vector

Step2: For simply laced groups, we have $gu = uu't$ with $u' \in U^+_{\geq 0}$, $t \in T_{>0}$

By step1. $\exists!$ $u \in U^-_{\geq 0}$ s.t. $g \in uB^+u^-$. We write $gu = uu't$ for some $u' \in U^+$. $t \in T$

But for $g \in G_{>0}$, we have $gu \in G_{>0} U^-_{\geq 0} \subseteq G_{>0} = U^-_{>0} U^+_{>0} T_{>0}$

Now in G, we have an iso $U^- \times U^+ \times T \xrightarrow{m} U^- U^+ T \subset G \Rightarrow uu't \in G_{>0}$ implies $u \in U^-_{>0}$.

$u' \in U^+_{\geq 0}$, $t \in T_{>0}$ similarly, if $g \in G_{w_1, w_2, >0}$, with $\mathrm{supp}(w_1) = \mathrm{supp}(w_2) = I$ (so that $g \in G^{osc}_{\geq 0}$)

$\Rightarrow gu \in G_{w_1, w_2, >0} U^-_{\geq 0} \subseteq U^-_{\geq 0} U^+_{w_1, >0} T_{>0}$. Thus $u' \in U^+_{w_1, >0} \subseteq U^+_{\geq 0}$, $t \in T_{>0}$

This finish the proof of existence when G is simply laced. The uniqueness follows by step1. u is unique.

Step3: From simply laced groups to nonsimply laced groups.

"folding method" Given any reductive group G. $\exists$ simply laced group $\dot{G}$ and a diagram automorphism σ s.t. $G = \dot{G}^\sigma$ $\quad G_{\geq 0} = \dot{G}^\sigma_{\geq 0}$, $\quad G^{osc}_{\geq 0} = (\dot{G}^{osc}_{\geq 0})^\sigma$. Let $g \in G^{osc}_{\geq 0} \subseteq \dot{G}^{osc}_{\geq 0} \cap \dot{G}^\sigma$

By step2, $\exists!\ u \in \dot{U}^-_{\geq 0}$, $u' \in \dot{U}^+_{\geq 0}$, $t \in \dot{T}_{>0}$ s.t. $g = uu'tu^{-1}$. Applying σ, we have $g = \sigma(g) = \sigma(u)\sigma(u')\sigma(t)\sigma(u^{-1})$, by the uniqueness in $\dot{G}$, $u, u', t \in \dot{G}^\sigma \Rightarrow u \in U^-_{\geq 0}$, $u' \in U^+_{\geq 0}$, $t \in T_{>0}$. This proves the existence of the decomposition in G. The uniqueness of the decomposition in G follows from the uniqueness in $\dot{G}$

Step4: In the decomposition, $\alpha_i(t) > 1$, $\forall i \in I$

proof: By the folding method, it suffices to consider the simply laced groups. In this case, $g = uu'tu^{-1}$ has algebraic multiplicity 1 for the maximum eigenvalue ($= \lambda(t)$ when $V = V\lambda$). alg multiplicity $1 \Rightarrow \alpha_i(t) \neq 1$, $\forall i \in I$.

Consider $T' = \{t \in T_{>0} : \alpha_i(t) \neq 1\}$. Then $G^{osc}_{\geq 0} \xrightarrow{\text{cont. map}} T'$. $G^{osc}_{\geq 0}$ is connected $\Rightarrow$ its image is connected.

$T_{>1} = \{t \in T_{>0} : \alpha_i(t) > 1\}$ is one connected component of T'.

Consider u_1, u_2, $u_1 \in U^-_{\geq 0}$, $u_2 \in U^+_{\geq 0}$, $u_1 u_2 = u u' t u^{-1}$. Let $u_2 \mapsto 1 \Rightarrow$ LHS $\to u_1$, RHS $\to$ $u_\varepsilon t_\varepsilon u_\varepsilon^{-1}$, $t_\varepsilon \mapsto 1$. $u_\varepsilon t_\varepsilon u_\varepsilon^{-1} \in U^-_{\geq 0} T_{>0} \Rightarrow \alpha_i(t_\varepsilon) > 1$

Example: $G = GL_2$ $\quad y(a) t y(-a) = y(a) y(-\alpha(t)^{-1} a) t = y(a - \alpha(t)^{-1} a) t \in U^-_{\geq 0} T_{>0} \Rightarrow \alpha(t) > 1$.

Question: How about the decomposition for other semifields?

Example: $G = GL_2$. $y(a) \alpha^\vee(b) x(c) \in G_{>0}$,

$$y(a)\alpha^\vee(b)x(c) = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} b & \\ & b^{-1} \end{pmatrix} \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$$

want: $gu = utu'$ $\quad y(a)\alpha^\vee(b) x(c) y(d) = g y(d) = y(d) \alpha^\vee(e) x(f)$

$$y(a)\alpha^\vee(b)x(c)y(d) = y(a) \alpha^\vee(b) y\left(\frac{d}{1+cd}\right) \alpha^\vee(1+cd) x\left(\frac{c}{1+cd}\right) = y(a) y\left(\frac{d}{(1+cd)b^2}\right) \alpha^\vee(b(1+cd)) x\left(\frac{c}{1+cd}\right)$$

So $d = a + \frac{d}{(1+cd)b^2}$, $e = b(1+cd)$, $f = \frac{c}{1+cd} \Rightarrow b^2 c d^2 + (b^2 - ab^2 c - 1) d - ab^2 = 0$. To solve d from a, b, c, one need to take square root $\Rightarrow$ such decomposition doesn't hold for any semifield.

Def. An ordered field is a field with a total order $>$, which is preserved under addition and multiplication by positive elements.

Example: $\mathbb{R}$ is an ordered field with total order given by $\mathbb{R}_{>0}$.

$\mathbb{R}((t))_{>0}$ universal semifield gives a total order on $\mathbb{R}((t))$.

Def. A real closed field is an ordered field s.t. 1. every positive has a square root
2. any polynomial of odd degree has a root.

Example: $\mathbb{R}$ is a real closed field, $\mathbb{R}((t))$ is not a real closed field.

Real Puiseux series $\mathbb{R}\{\{t\}\} = \bigcup_{n \in \mathbb{N}} \mathbb{R}((t^{\frac{1}{n}}))$ is a real closed field.

$\mathbb{R}\{\{t\}\} + \sqrt{-1}\,\mathbb{R}\{\{t\}\} = \mathbb{C}\{\{t\}\}$ is algebraically closed and complete.

Eaves-Rothblum-Schneider: Perron-Frobenius Theory over real closed fields and fractional power series expansions

In which they generalize Perron-Frobenius theory via logic.

Upshot (?): It is likely that the decomposition (*) holds for real closed fields, and for Trop $\mathbb{Q}$.

Here (Trop $\mathbb{Q}$, $\oplus$, $\odot$), $a \oplus b = \min(a,b)$, $a \odot b = a+b$

Back to Gl_2 example: $d = a + \frac{d}{(1+cd)b^2}$. in Trop $\mathbb{Q}$ it becomes $d = \min(a,\ d - 2b - \min(1, c+d))$ (*)

For any a, b, c, $\exists d$ satisfying $(*)$ in $\mathbb{Q}$ or in $\mathbb{Z}$.

Question: Whether the decomposition holds over Trop $\mathbb{Z}$?

Gl_2, Gl_3 can be checked directly