In this week, we study the interplay between total positivity and conjugacy classes of groups. First, let us recall the abstract Jordan decomposition.

Definition 7.1. *Let G be a connected complex reductive group and $g \in G$. Then,*

- *the element g is **semisimple** if for all representations (ρ, V) of G such that $\rho(g) \in \operatorname{Aut}(V)$ is diagonalizable,*
- *the element g is **unipotent** if for all representations (ρ, V) of G such that $\rho(g) \in \operatorname{Aut}(V)$ is unipotent.*

Theorem 7.2 (Abstract Jordan Decomposition)**.** *Let G be a connected complex reductive group and $g \in G$. Then, there exists an unique pair of elements $g_s, g_u \in G$ such that*

1. *the element g_s is semisimple,*
2. *the element g_u is unipotent,*
3. *we can express g as $g = g_s g_u = g_u g_s$.*

Example 7.3. *Let us illustrate this example in $G = \mathrm{GL}_n$. This is just the Jordan Canonical Form. For any matrix $M \in \mathrm{GL}_n(\mathbb{C})$, we know that it is conjugate to*

$$M \cong \begin{pmatrix} J_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & J_2 & 0 & \dots & 0 & 0 \\ 0 & 0 & J_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & J_{k-1} & 0 \\ 0 & 0 & 0 & \dots & 0 & J_k \end{pmatrix}$$

where J_k are the Jordan blocks. But every Jordan block

$$J_i \cong \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda & 1 & \dots & 0 & 0 \\ 0 & 0 & \lambda & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda & 0 & 0 & \dots & 0 & 0 \\ 0 & \lambda & 0 & \dots & 0 & 0 \\ 0 & 0 & \lambda & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda & 0 \\ 0 & 0 & 0 & \dots & 0 & \lambda \end{pmatrix} \begin{pmatrix} 1 & \lambda^{-1} & 0 & \dots & 0 & 0 \\ 0 & 1 & \lambda^{-1} & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & \lambda^{-1} \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

as a product of a semisimple part and a unipotent element.

Example 7.4. *We also have that elements in T are semisimple and elements in $U^{\pm}$ are unipotent. Moreover, over $\mathbb{C}$, any semisimple element in (connected) G is conjugate to some element in T by an element of $W = N(T)/T$ and any unipotent element is conjugate to some element in $U^{\pm}$.*

Definition 7.5. *We also can define the Steinberg map $\mathrm{St} : G \to G//G \cong T/W$ where $G//H$ is the GIT quotient by sending g to the conjugacy class of g_s (as Theorem 7.2).*

Next, we can define regular elements.

Lemma 7.6. *Let $g \in G$ be an element, then the following are equivalent.*

- *The centralizer* $\dim C_G(g) = \mathrm{rk}(G)$,
- *The centralizer* $\dim C_G(g)$ *is minimal,*

Proof. See [Humphreys' Conjugacy Classes in Semisimple Algebraic Groups §1.6, 2.3] ☐

Definition 7.7. *An element $g \in G$ satisfying any of the properties in Theorem 7.6 is said to be regular.*

Example 7.8. *For any element $t \in T$, t is regular if and only if $\alpha(t) \neq 1$ for all roots α. For any element $u \in U^-$, if the map γ in Lemma 6.1 has all its coordinates non-zero. In particular, for $G = \mathrm{GL}_n$, where*

$$t = \begin{pmatrix} a_1 & 0 & ... & 0 \\ 0 & a_2 & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & a_n \end{pmatrix}$$

we require $a_i = a_j$ if and only if $i = j$. Similarly,

$$t = \begin{pmatrix} 1 & b_1 & ... & * \\ 0 & 1 & ... & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & b_{n-1} \\ 0 & 0 & ... & 1 \end{pmatrix}$$

we require $b_i \neq 0$ for all $i \in \{1, 2, ..., n-1\}$.

Theorem 7.9. *The Steinberg map is a well-defined bijection from the regular conjugacy classes (conjugacy classes of regular elements) to the semisimple conjugacy classes.*

Proof. See [Humphreys' Conjugacy Classes in Semisimple Algebraic Groups §4.10] ☐

Theorem 7.10. *For any connected reductive group, there is a unique regular unipotent conjugacy class.*

Proof. [Humphreys' Conjugacy Classes in Semisimple Algebraic Groups §4.5] shows that such an element must exist, while [Humphreys' Conjugacy Classes in Semisimple Algebraic Groups §4.6] shows that all regular unipotent elements are conjugate to one another. ☐

Definition 7.11. *Let $g \in G_{\geq 0}$ be an element. We say that g is oscillatary if $g^m \in G_{>0}$ for some $m \in \mathbb{Z}_{>0}$. Moreover, we denote the set of oscillatary elements in $G_{\geq 0}$ as $G_{\geq 0}^{osc}$.*

To understand more precisely what oscillatary elements are, we would have to define the notion of "support" of an element in the Weyl group.

Theorem 7.12. *Let $w \in W$ be an element in the Weyl group. Moreover, let s be a simple reflection appearing in some reduced expression of w. Then, for any reduced expression of w, s would occur.*

Proof. This is as the only relations to be applied are braid relations. (Quadratic relations would reduce the length, contradicting the minimality of the reduced expression) However, braid relations do not change the occurence of simple reflections. ☐

Remark 7.13. *For any $w \in W$ as the above theorem, we define* $\mathrm{supp}(s)$ *to be the set of simple reflections occuring in any (equivalently all) reduced expressions of w.*

Theorem 7.14. *The set of oscillatory elements can be decomposed as*

$$G^{osc}_{\geq 0} = \bigsqcup_{\substack{u,v \in W \\ \mathrm{supp}(u)=\mathrm{supp}(v)=I}} G_{u,v,>0}$$

Proof. Let $u, v \in W$ and $g \in G_{u,v,>0}$. Then,

$$\begin{aligned} g^2 &\in G_{u,v,>0} G_{u,v,>0} \\ &= G_{u*u, v*v, >0} \end{aligned}$$

where $*$ is the monoidal product. We denote u^{*n} as $u^{*n-1} * u$ when $n > 1$ and u when $n = 1$. Then, we have

$$g^n \in G_{u^{*n}, v^{*n}, >0}.$$

Here, we also note that $\mathrm{supp}(\alpha * \beta) = \mathrm{supp}(\alpha) \cup \mathrm{supp}(\beta)$. Hence, $\mathrm{supp}(u^{*n}) = \mathrm{supp}(u)$ and similarly $\mathrm{supp}(v^{*n}) = \mathrm{supp}(v)$. Now, we further suppose that g is oscillatary with $g^n \in G_{>0} = G_{w_0, w_0, >0}$. Then,

$$\mathrm{supp}(u) = \mathrm{supp}(v) = \mathrm{supp}(w_0) = I$$

where w_0 is the longest element, and thus would consist of all the simple reflections.
Conversely, there exists an $m \in \mathbb{Z}_{\geq 0}$ such that for any u satisfying $\mathrm{supp}(u) = I$, then $u^{*m} = w_0$. Thus, for any $g \in G_{u,v,>0}$, $g^m \in G_{w_0,w_0,>0} = G_{>0}$. □

7.1 Oscillatory Decomposition Lemma

The next lemma pertains to a decomposition of oscillatory elements.

Lemma 7.15 (Oscillatory Decomposition)**.** *Let $\sigma \in G^{osc}_{\geq 0}$ be an oscillatory element. Then, there exists unique $u \in U^-_{>0}, u' \in U^+_{>0}$ and $t \in T_{>0}$, such that $\sigma u = uu't$. Moreover, for any simple root α, $\alpha(t) > 1$.*

Remark 7.16. *The proof of this statement relies on the theory of Canonical Basis, as well as the flag variety, and would be deferred to a later part in the course.*

Remark 7.17. *It is an open question to determine if such a decomposition can hold over a general semifield $\Bbbk$.*

Example 7.18. *Let us illustrate Theorem 7.15 in the case where $G = \mathrm{GL}_2$ and a special choice of oscillatory element.*
We note that the elements in the torus of the form

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \,\middle|\, a > b > 0 \right\}$$

are oscillatory elements. (The condition $a > b$ is such that the only simple root α acting on this element would give us $a/b > 1$.)
Then, if we pick

$$u^{-1} = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \in U^-_{>0}$$

, where $c > 0$, we have

$$\begin{aligned} u^{-1} t u &= \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ c(1 - \frac{b}{a}) & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in U^-_{>0} T_{>0} \end{aligned}$$

For the rest of this lecture, we shall see various applications of this lemma.
We shall denote G^{reg} (resp. G^{uni} or $G^{\text{reg, s.s}}$) as the set of regular (resp. unipotent or regular semisimple) elements in G.

Theorem 7.19. *Let $w_1, w_2 \in W$. We have that*

1. *each totally positive cell $G_{w_1,w_2,>0}$ in G contains a regular semisimple element. (i.e. $G_{w_1,w_2,>0} \cap G^{reg,\ s.s} \neq \emptyset$.)*
2. *the totally positive cell $G_{w_1,w_2,>0}$ lies in $G^{reg,\ s.s}$ if and only if* $\operatorname{supp}(w_1) = \operatorname{supp}(w_2) = I$.
3. *the decomposition*
$$G^{uni} \cap G_{\geq 0} = \bigsqcup_{\substack{u,v \in W \\ \operatorname{supp}(u)\ \cap\ \operatorname{supp}(v) = \emptyset}} U^+_{u,>0} U^-_{w_2,>0}.$$

Proof. Let $\operatorname{supp}(w_i) = J_i$ for $i \in \{1, 2\}$.
Step 1: We reduce to the case where $J_1 \cap J_2 = \emptyset$.
Let $k = J_1 \cap J_2$. The idea here is that we have the "overlap" can be encapsulated in the Levi subgroup corresponding to k, L_k. Thus, for any $\sigma = u_1 t u_2$ where $u_1 \in U^-_{w_1,>0}$, $t \in T_{>0}$ and $u_2 \in U^+_{w_2,>0}$, we will be able to factor

$$\begin{aligned} u_1 &= u_1'' u_1' \text{ such that} \\ u_1' &\in U^- \cap L_k \\ u_1'' &\in U^- \cap R_u(P_k^-) \end{aligned}$$

where P_k is the parabolic subgroup corresponding to k and $R_u(H)$ refers to the unipotent radical of H. Similarly,

$$\begin{aligned} u_2 &= u_2' u_2'' \text{ such that} \\ u_2' &\in U^+ \cap L_k \\ u_2'' &\in U^+ \cap R_u(P_k^+) \end{aligned}$$

But, we have that $u_1' t u_2'$ is not just an element of L_k but an oscillatory element of $(L_k)_{\geq 0}$. Hence, by Theorem 7.15, we have $u^- \in U^-_{k,>0}$, $u^+ \in U^+_{k,>0}$ and $t' \in T_{\geq 0}$ satisfying

$$\begin{aligned} (u_1' t u_2') u^- &= u^- u^+ t' \\ (u_1' t u_2') &= u^- u^+ t' (u^-)^{-1} \\ \alpha_i(t') &> 1 \text{ for all } i \in k. \end{aligned}$$

The second line here shows that after conjugation by u^- we are only left with the positive part. Thus,

$$\begin{aligned} (u^-)^{-1} \sigma u^- &= (u^-)^{-1} u_1 t u_2 u^- \\ &= (u^-)^{-1} u_1'' u_1' t u_2' u_2'' u^- \\ &\in (L_{J_1} \cap R_u(P_k^-))((u^-)^{-1} u_1' t u_2' u^-)(L_{J_2} \cap R_u(P_k^+)) \\ &= (L_{J_1} \cap R_u(P_k^-))(u^+ t')(L_{J_2} \cap R_u(P_k^+)) \end{aligned}$$

The upshot is we can find corresponding changes for u_1, u_2 when we change t to t'.
Step 2: In this step we specify t such that the t' constructed from Step 1 satisfy $\alpha_i(t') > 1$ for $i \in I \setminus k$. This is possible as $(L_{J_1} \cap R_u(P_k^-))$ and $(L_{J_2} \cap R_u(P_k^+))$ commute with conjugation of the centralizer of L_k. In particular, this means that t' is regular semisimple in T. Thus, there exists a σ such that it can be conjugated to yield t', a regular semisimple element. Hence, part (1) is shown.

Step 3: Next we shall show part (2).
If $\mathrm{supp}(w_1) = \mathrm{supp}(w_2) = I$, then by Theorem 7.15, we have $G_{w_1,w_2,>0} \subseteq G_{\geq 0}^{\text{reg, s.s}}$. Conversely, if $k \neq I$, then there exists σ such that the t' constructed in Step 1 satisfy $\alpha_i(t') \leq 1$ for $i \in k$. But this means that the elements in $(L_{J_1} \cap R_u(P_k^-))(u^+t')(L_{J_2} \cap R_u(P_k^+))$ are not regular semisimple, which contradict our assumptions on σ. $\square$

The proof of part (3) is deferred to the following lecture.

Remark 7.20. *For part (2) in Theorem 7.19, this can be relaxed to $G_{w_1,w_2,>0}$ lies in G^{reg} if and only if* $\mathrm{supp}(w_1) = I$ *or* $\mathrm{supp}(w_2) = I$. *This is the main result of [He-Lusztig Total Positivity and Conjugacy Classes].*

Corollary 7.21. *Let $\mathring{G}_{\geq 0}^{reg,\ s.s}$ (resp. $\mathring{G}_{\geq 0}^{reg}$) be the union of all the cells of $G_{\geq 0}$ lying entirely in $G^{reg,\ s.s}$ (resp. G^{reg}). Then, $\mathring{G}_{\geq 0}^{reg,\ s.s} = G_{\geq 0}^{osc}$ is an open subgroup of $G_{\geq 0}$ and $\mathring{G}_{\geq 0}^{reg}$ is also an open subgroup of $G_{\geq 0}$.*
Moreover, they are also two-sided ideals.

$$G_{\geq 0}\mathring{G}_{\geq 0}^{reg,\ s.s}, \mathring{G}_{\geq 0}^{reg,\ s.s}G_{\geq 0} \subseteq \mathring{G}_{\geq 0}^{reg,\ s.s}$$
$$G_{\geq 0}\mathring{G}_{\geq 0}^{reg}, \mathring{G}_{\geq 0}^{reg}G_{\geq 0} \subseteq \mathring{G}_{\geq 0}^{reg}$$