Week 3.

Last time: Weyl gp.

Length function
- (i) via reduced expressions, generators of order 2.
- (ii) via root system $|\Phi^+ \cap w(\Phi^-)|$.

Exchange property: If $w = s_{i_1} \cdots s_{i_n}$ reduced. $\ell(sw) < \ell(w)$ then
$$sw = s_{i_1} \cdots \hat{s_{i_j}} \cdots s_{i_n}.$$

Characterization: For any gp gen. by ele. of order 2. It is a coxeter gp iff the exchange property holds.

Word problem: If $w = s_{i_1} \cdots s_{i_n}$ reduced
$= s_{j_1} \cdots s_{j_n}$ reduced

then $s_{i_1} \cdots s_{i_n} = s_{j_1} \cdots s_{j_n}$ in <u>braid gps</u>.

(A <u>braid gp</u> is coxeter gp without $s_i^2 = 1$ relation and only with coxeter relations).

<u>Bruhat decomposition</u>

(Ref. Kumar's Kac-Moody gps, their flag varieties and rep. theory.

Def. (Tits system).

A <u>Tits system</u> (or BN-pairs) is (G, B, N, S) where

G is a gp

$B, N \leq G$ subgps s.t. $B \cap N \trianglelefteq N$ normal.

S: $S \subseteq \underset{\text{Weyl gp}}{W} := N/B\cap N$ a finite subset of

s.t.

(BN1) S generates W. (S big enough)

(BN2) B, N generates G. (B, N big enough).

(BN3) $\forall s \in S$, $sBs^{-1} \neq B$ (no redundency for s).

(BN4) $C(s)C(w) \subseteq C(w) \cup C(sw)$ $\forall s \in S$, $w \in W$.

where $C(w) = B\dot{w}B$.

We usually use $\dot{w} \in N$ as a representative.

Proof of Bruhat Decomposition.

Thm. Let (G,B,N,S) be a Tits system. Then.

(i). $G = \coprod_{w\in W} C(w)$ is a disjoint union.

(ii). (W,S) is a coxeter group.

pf. ① Claim: $s^2=1 \quad \forall s\in S$.

pf of ①: $C(s)C(s^{-1}) \subseteq C(s^{-1}) \cup C(1)$. (BN4)

Taking inverse on both sides

$$\underbrace{C(s^{-1})^{-1}}_{= C(s)} \underbrace{C(s)^{-1}}_{= C(s^{-1})} \subseteq C(s) \cup C(1).$$

①-1 Claim. $C(s)C(s^{-1}) = C(s) \amalg C(1)$.

pf of ①-1:

First, $C(s)\cap C(1) = \phi$.

Otherwise, $C(s)\cap C(1) \neq 0$

$\Rightarrow C(s) = C(1)$

$\Rightarrow \dot{s} = 1$ in W

$\Rightarrow\Leftarrow$ to (BN3).

Second, we show this is equality.

Indeed, $C(1) = Bss^{-1}B \subseteq C(s)C(s^{-1})$

$C(s)C(s^{-1}) = \underbrace{BsBs^{-1}B}_{\not\subseteq B} \not\subseteq B = C(1)$!

by (BN3).

As $C(s)C(s^{-1})$ is a union of double cosets, it must be the union of two cosets $C(s) \amalg C(1)$. $\square$ ①-1.

Similarly, $C(s)C(s^{-1}) = C(s^{-1})C(1)$.

Hence $C(s) = C(s^{-1})$.

Now by (BN4), taking $w=s$,

$C(s)C(s) \subseteq C(s) \cup C(s^2)$.

and together with previous results,

$C(s) \amalg C(1) \subseteq C(s) \cup C(s^2)$.

Hence $C(s^2) = C(1)$.

Then $(\dot{s})^2 \in B\cap N$. Hence $s^2 = 1$ in W. $\square$ ①.

②. $G = \bigcup_{w\in W} C(w)$.

pf of ②. Set $G' = \bigcup_{w\in W} C(w)$.

By (BN1) and (BN2),

G is generated by B and $\dot{s}$, $s\in S$.

Therefore, to prove $G\subseteq G'$ it suffices to show

- $B\cdot G' \subseteq G'$ and
- $\dot{s}\cdot G' \subseteq G'$

The first one is obvious since G' is union of double B-cosets. For the second, by (BN4)

$\dot{s}\, C(w) \subseteq C(s)\cdot C(w) \subseteq C(w) \cup C(sw) \subseteq G'$. $\square$ ②.

③ $G = \coprod_{w\in W} C(w)$ is disjoint union, i.e., if $C(v) = C(w)$ then $v=w$, for $v,w\in W$.

pf of ③. By part ①, length is well-defined.

Assume $\ell(w) \le \ell(v)$.

If $\ell(w)=0$, then $w=1$. $C(1) = C(v) \Rightarrow v=1$.

If $\ell(w)>0$, then $\exists\, s\in S$ s.t.

$w = sw'$ and $\ell(w) = \ell(w')+1$.

$w'B \subseteq sC(w) = sC(v) \subseteq C(v) \cup C(sv)$.

Hence $C(w') \subseteq C(v) \cup C(sv)$.

If $C(w') = C(v)$ then by induction on length

$\ell(v) = \ell(w') < \ell(w)$. $\Rightarrow\Leftarrow$.

If $C(w') = C(sv)$, then by induction on length

$w' = sv$ and so $w = sw' = v$. $\square$ ③.

④ (BN4) is actually in two cases.

$$C(s)C(w) = \begin{cases} C(sw) & \text{if } \ell(sw) \ge \ell(w). \\ C(w)\cup C(sw) & \text{if } \ell(sw) < \ell(w). \end{cases}$$

④-1. $C(s)C(w) = C(sw)$ if $\ell(sw) \ge \ell(w)$.

pf of ④-1. If $\ell(w)=0$ this is obvious.

If $\ell(w)>0$, write $w = w't$ for some $t\in S$ and $\ell(w) = \ell(w')+1$.

By induction, $C(t)C(w'^{-1}) = C(w^{-1})$.

Thus $C(w) = C(w')C(t)$.

Now $C(s)C(w')C(t) = C(s)C(w) \subseteq C(w) \cup C(sw)$.

If $C(s)C(w) \neq C(sw)$. then

$$C(s)C(w) \cap C(w) \neq \phi$$

$\Rightarrow sBw \cap C(w) \neq \phi$.

$\Rightarrow sBw' \cap C(w)t \neq \phi$ (#1).

As $\ell(sw) \geq \ell(w)$, $\ell(sw') \geq \ell(w')$.

$\Rightarrow sBw' \subseteq C(sw')$ by induction (#2)

Also $C(w)t \subseteq C(w) \cup C(w')$. (BN4).

We know $sw' \neq w'$, so $C(sw') \cap C(w') = \phi$.

We claim $sw' \neq w$ also, for otherwise,

$\ell(w') = \ell(sw) \geq \ell(w)$ $\Rightarrow\Leftarrow$.

so $C(sw') \cap C(w') = \phi$.

$\Rightarrow C(sw') \cap (C(w) \cup C(w')) = \phi$.

$\Rightarrow C(sw') \cap C(w)t = \phi$

$\overset{(\#2)}{\Rightarrow} C(w)t \cap sBw' = \phi$. $\Rightarrow\Leftarrow$ with (#1). □ ④-1

(④-2 cont'd on right hand side).

Remark. ④ has a variation

④'
$$C(w)C(s) = \begin{cases} C(ws) & \text{if } \ell(ws) \geq \ell(w) \\ C(w) \amalg C(ws) & \text{if } \ell(ws) < \ell(w). \end{cases}$$

(take the inverse of ④).

⑤ (W, S) satisfies exchange property, i.e.,

for $w = s_{i_1} \cdots s_{i_n}$ a reduced expression, $s \in S$.

Assume $\ell(sw) < \ell(w)$. We want to show

$$sw = s_{i_1} \cdots \hat{s}_{i_j} \cdots s_{i_n}$$

pf of ⑤.

By ④, $C(w) = C(s_{i_1}) \cdots C(s_{i_n})$.

so $C(s)C(w) = C(s)C(s_{i_1}) \cdots C(s_{i_n})$

By ④, the left-hand side

$$C(s)C(w) = C(w) \amalg C(sw)$$
$$= C(w) \amalg C(\underbrace{s s_{i_1} \cdots s_{i_n}}_{\text{full sequence}}).$$

By (BN4), the right hand side

$$C(s)C(s_{i_1}) \cdots C(s_{i_n}) \subseteq \bigcup_{\substack{1 \leq j_1 < j_2 < \cdots < j_\ell \leq n \\ \text{all subsequences}}} C(s s_{i_{j_1}} \cdots s_{i_{j_\ell}})$$

(Note that this is not disjoint as different j's may give same ele in W).

Thus, $C(w) = C(s s_{i_{j_1}} \cdots s_{i_{j_\ell}})$ for some proper subsequence

then $w = s s_{i_{j_1}} \cdots s_{i_{j_\ell}}$ by ③.

As $\ell(w) = n$, $\ell(s s_{i_{j_1}} \cdots s_{i_{j_\ell}}) \leq 1 + \underset{\text{length of subseq.}}{(n-1)} = n$.

We see

$$w = s \underbrace{s_{i_{j_1}} \cdots s_{i_{j_\ell}}}_{\text{length } n-1} \text{ is a reduced expression}$$

where $s_{i_{j_1}} \cdots s_{i_{j_\ell}} = s_{i_1} \cdots \hat{s}_{i_j} \cdots s_{i_n}$.

so $sw = s_{i_1} \cdots \hat{s}_{i_j} \cdots s_{i_n}$

□ ⑤.

□ Thm.

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④-2. $C(s)C(w) = C(w) \cup C(sw)$ if $\ell(sw) < \ell(w)$.

pf of ④-2.

Write $w' = sw$. $\ell(sw') > \ell(w')$.

By ④-1 we have

$C(s)C(w') = C(w)$

so $C(s)C(w) = C(s)C(s)C(w')$

$\overset{①}{=} (C(s) \amalg C(1))C(w')$

$= C(s)C(w') \cup C(1)C(w')$.

$= C(w) \cup C(w')$.

□ ④-2

□ ④.

Remark. $G = \coprod_{w \in W} C(w)$ is called <u>Bruhat decomposition</u>.

Eg. $G = GL_n(\mathbb{F})$. $\mathbb{F}$ any field.

$$B = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\} \subseteq G \quad \text{Borel subgp}$$

$$T = \left\{ \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\} \in G \quad \text{maximal torus.}$$

$N = N_G(T)$ consists of matrices with exactly one non-zero entry in each row and each column.

Then $T = B \cap N$. $W = N/T \cong S_n$.

$$S = \{ s_1 = (12), \dots, s_{n-1} = (n-1, n) \} \subseteq W.$$

Thm: (G, B, N, S) is a Tits system.

pf. (BN1)–(BN3) is easy.

(BN4): $C(s)\,C(w) \subseteq C(w)\,C(sw)$

Since we already know (W, S) is a coxeter gp. It suffices to prove

① $C(s)\,C(s) = C(s) \sqcup C(1) \quad \forall s \in S.$

$$\dot{s}_i = \begin{pmatrix} I & & \\ & \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} & \\ & & I \end{pmatrix}$$

$(i, i+1)$-position

$(i+1, i)$-position

This is calculation of GL_2.

② If $\ell(sw) > \ell(w)$ then

$$C(s)\,C(w) = C(sw).$$

To prove this, we want to

$$BsBwB \overset{?}{=} BswB$$

move the middle B to left and right.

Decompose $B = UT$.

$$U = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\}.$$

$BsUTwB = BsUwTB$ as $w \in N_G(T)$.

We claim U can be moved to the sides by quoting the following

Fact: $U = \prod_{\alpha \in \Phi^+} U_\alpha$ for any given order on Φ^+.

root subgp

In case of GL_n, $U_{e_i - e_j} = \left\{ \begin{pmatrix} 1 & & *\,0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\}$ (i,j)-pos.

Pf (cont'd)

Now choose an order on Φ^+ so that γ occurs in the end, where $s = s_\gamma$ simple reflection.

Then

$$U = \prod_{\substack{\alpha \in \Phi^+ \\ \alpha \neq \gamma}} U_\alpha \, U_\gamma$$

As $s_\gamma(\alpha) \in \Phi^+$, $\forall \alpha \neq \gamma$,

$$Bs \Big(\prod_{\substack{\alpha \in \Phi^+ \\ \alpha \neq \gamma}} U_\alpha \Big) = B \prod_{\substack{\alpha \in \Phi^+ \\ \alpha \neq \gamma}} U_{s_\gamma(\alpha)} s_\gamma = B s_\gamma .$$

($t U_\alpha t^{-1} = U_{t(\alpha)}$) c.f. 8.1.12(2) of Springer's alg. gps

Since $\ell(sw) > \ell(w)$ we have

$$w^{-1}(\gamma) \in \Phi^+$$

so

$$U_\gamma w B = w U_{w^{-1}(\gamma)} B = wB$$

Thus

$$B s_\gamma U w B = B s_\gamma w B = C(s_\gamma w) \quad \square$$

(BN4)

$\square$

Thm

Remark. Similar construction works for any reductive gps split over $\mathbb{F}$.

Def. (flag variety, Schubert variety)

Let G be a reductive gp split over $\mathbb{F}$.

We have Bruhat decomposition

$$G = \coprod_{w\in W} BwB. \quad B \subseteq G \text{ Borel subgp}/\mathbb{F}.$$

In particular,

(i) $G/B = \coprod_{w\in W} BwB/B$ called <u>flag variety</u>

(ii) Each piece BwB/B is called <u>schubert cell</u>

(iii) Over any alg. closed field, the closure

$$\overline{BwB/B}$$

is called <u>schubert variety</u>.

Remark. In general, schubert varieties are singular, and their singularity play a crucial role in representation theory (Kazhdan-Lusztig theory).

<u>Resolution of singularities</u> (Bott-Samelson variety).

Def. (minimal parabolic subgp). Assume $\mathbb{F} = \overline{\mathbb{F}}$.

Let (G,B,N,S) be a Bruhat decomposition (of Kac-Moody gps).

For any $s\in S$. Define the <u>minimal parabolic subgroup</u> corresponding to s:

$$P_s := C(s) \amalg C(1)$$
$$(= B W_{\{s\}} B)$$

This is a group by ① of pf of Bruhat decomposition).

Remark. $P_s/B \cong \mathbb{P}^1$

$C(s)/B \cong \mathbb{A}^1$

$C(1)/B \cong pt.$

Construction of Bott-Samelson variety:

Let $w = s_{i_1}\cdots s_{i_n}$ be a reduced expression.

Consider $P_{i_1}\times P_{i_2}\times\cdots\times P_{i_n} \circlearrowleft B^n$

with action defined by

$$(b_1,\dots,b_n)\cdot(P_1,\dots,P_n) = (P_1 b_1^{-1}, b_1 P_2 b_2^{-1}, \dots, b_{n-1} P_n b_n).$$

Let $P_{i_1}\times^B P_{i_2}\times^B\cdots\times^B (P_{i_n}/B)$

then we have a well-defined map

$$\pi: P_{i_1}\times^B\cdots\times^B (P_{i_n}/B) \xrightarrow{\pi} G/B$$
$$(P_1,\dots,P_n) \longmapsto P_1P_2\cdots P_n B/B.$$

π is proper over its image.

As a consequence,

$$\mathrm{Im}\,\pi = \overline{BwB/B}$$

Now

$$\begin{tikzcd}
C(s_{i_1})\times^B\cdots\times^B C(s_{i_n})/B \arrow[r, "④", "\cong"'] \arrow[d, "\text{open dense}"] & C(w)/B \arrow[d] \\
P_{i_1}\times^B\cdots\times P_{i_n}/B \arrow[r, "\text{resolution of singularities}"'] & \overline{BwB/B}
\end{tikzcd}$$

Define $BS(\underline{w}) = P_{i_1}\times^B\cdots\times P_{i_n}/B$

$\underline{w} = s_{i_1}\cdots s_{i_n}$ reduced exp.

called the <u>Bott-Samelson variety</u>

It is a smooth variety as it is

$$\underbrace{\mathbb{P}^1 \text{ over } \mathbb{P}^1 \text{ over } \cdots \text{ over } \mathbb{P}^1}_{n \text{ times}}.$$

Another consequence is

$$\mathrm{Im}(\pi) = \underbrace{\bigcup_{1\le j_1<\cdots<j_n\le n} B s_{i_{j_1}}\cdots s_{i_{j_n}} B/B}_{\text{indep. of choice of reduced exp.}}$$

<u>Bruhat order</u>

Thm: Let $w\in W$, $w'\in W$. The followings are equivalent.

(i). $w' = s_{i_{j_1}}\cdots s_{i_{j_n}}$ for some subexpression of a reduced exp. $w = s_{i_1}\cdots s_{i_n}$ of w.

(ii) w' can be realized as subexp of any reduced exp. of w

Def. (Bruhat order)

In the above case, we write $w' \leq w$ called <u>Bruhat order</u>.

Thm.

$$\overline{C(w)} = \coprod_{w' \leq w} C(w') \qquad (\mathbb{F} = \overline{\mathbb{F}}).$$